Oscillation Criteria of Even-Order Nonlinear Difference Equations with a Sub-linear Neutral Term via Comparison Method

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Abstract

In this paper, the authors present some new oscillation criteria for the nonlinear difference equation

$$\Delta^m \left(x_n + p_n x_{n-k}^{\alpha}\right) + q_n x_{n-\ell}^{\beta} = 0$$

by using comparison method. Examples are provided to illustrate the significance of the main results.

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1 Introduction

In this paper, we study the oscillatory behavior of even order nonlinear neutral difference equation

$$\Delta^m \left(x_n + p_n x_{n-k}^{\alpha}\right) + q_n x_{n-\ell}^{\beta} = 0, \quad n \in \mathbb{N}(n_0) \tag{1.1}$$

where $\mathbb{N}(n_0) = \{n_0,\ n_0+1,\ \ldots\}$, n_0 is a positive integer, subject to the following hypothesis:

(H_1) $m \geq 2$ is an even integer, and α and β are ratio of odd integers with $0 < \alpha \leq 1$ and $\beta \in (0, \infty)$;

(H_2) $\{p_n\}$ and $\{q_n\}$ are positive real sequences for all $n \in \mathbb{N}(n_0)$ with $\lim_{n\to\infty} p_n = 0$;

(H_3) ℓ and k are positive integers.

By the solution of equation (1.1), we mean a real valued sequence that satisfies equation (1.1) for all $n \in \mathbb{N}(n_0)$. We consider only those solutions $\{x_n\}$ of equation (1.1) that satisfy $\sup\{|x_n| : n \geq N\} > 0$ for all $N \in \mathbb{N}(n_0)$ and tacitly assume that equation (1.1) has such solutions. Such a solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; else it is nonoscillatory. If all solutions are oscillatory then equation (1.1) itself called oscillatory.

The neutral type difference equations naturally arises in the applications including problems in population dynamics, in cobweb models in economics or in the design of high speed computer lossless transmission lines and so on. Therefore the problem of investigating the oscillatory behavior of solutions of neutral type difference equations has been studied by many authors; see for examples [1, 2, 4, 5, 8, 10–13, 18] and the references cited therein. In many results the neutral term is linear and few results are available when the neutral term is nonlinear; see [3, 6, 7, 9, 14–17].

In this paper, we shall present some new criteria for the oscillation of all solutions via a comparison with higher order linear delay difference inequalities as well as with first order delay difference equations whose oscillatory characters are known,. Examples are provided to illustrate the importance of the main results.

2 Main Results

We begin with the following lemmas that are important in the proofs of our main results.

Lemma 2.1. *(Discrete Kneser's Theorem) Let $\{u_n\}$ be a positive real sequence with $\Delta^m u_n$ being of constant sign eventually and not identically zero eventually. Then there exists an integer j, $0 \le j \le m$, with $(m+j)$ odd for $\Delta^m u_n \le 0$ and $(m+j)$ even for $\Delta^m u_n \ge 0$ and $N \in \mathbb{N}(n_0)$ such that*

$$\Delta^i u_n > 0 \ \ \textit{for all} \ \ i = 1, \ 2, \ . \ . \ . \ , \ j$$

and

$$(-1)^{i+j}\Delta^i u_n > 0 \ \ \textit{for all} \ \ i = j+1, \ j+2, \ . \ . \ . \ , \ m-1$$

for all $n \ge N$.

Lemma 2.2. *Let $\{u_n\}$ be a positive real sequence with $\Delta^m u_n \le 0$ for all $n \in \mathbb{N}(n_0)$ and not identically zero. Then there is an integer $n_1 \in \mathbb{N}(n_0)$ such that*

$$u_n \ge \frac{(n-n_1)^{m-1}}{(m-1)!}\Delta^{m-1}u_{2^{m-j-1}n}, \ n \ge n_1$$

where j is defined in Lemma 2.1. Further, if $\{u_n\}$ is increasing, then

$$u_n \ge \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1}u_n, \quad n \ge 2^{m-1}n_1.$$

The proof of the last two lemmas can be found in [1].

Before starting the next lemma, we define

$$z_n = x_n + p_n x^{\alpha}_{n-k}.$$

Lemma 2.3. *Let $\{x_n\}$ be a positive solution of equation* (1.1). *Then there is an integer $n_1 \in \mathbb{N}(n_0)$ such that*

$$z_n > 0, \ \Delta z_n > 0, \ \Delta^{m-1} z_n > 0 \ and \Delta^m z_n \leq 0$$

for all $n \geq n_1$.

Proof. The proof is similar to that of Lemma 3 of [12], and hence it is omitted. □

Theorem 2.4. *Let $\beta > 1$. If the even-order linear delay difference inequality*

$$\Delta^m z_n + M q_n z_{n-\ell} \leq 0 \tag{2.1}$$

has no positive solution for every constant $M > 0$, then equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1), say $x_n > 0, \ x_{n-k} > 0$ and $x_{n-\ell} > 0$ for all $n \geq n_1 \in \mathbb{N}(n_0)$ since the proof for the negative case is similar. From equation (1.1) and (H_3), we have

$$\Delta^m z_n = -q_n x^{\beta}_{n-\ell} < 0, \quad n \geq n_1. \tag{2.2}$$

Then, it follows from Lemma 2.1, there is an integer $n_2 \geq n_1$ such that

$$\Delta z_n > 0 \ \text{ and } \ \Delta^{m-1} z_n > 0 \tag{2.3}$$

for all $n \geq n_2$. From the definition of z_n, we obtain

$$x_n = z_n - p_n x^{\alpha}_{n-\ell} \geq z_n - p_n z^{\alpha}_n \tag{2.4}$$

where we have used $z_n \geq x_n$ and $\{z_n\}$ is increasing. Since z_n is positive and increasing, there is an integer $n_2 \geq n_1$ and a constant $c > 0$ such that

$$z_n \geq c \ \text{ for all } \ n \geq n_2. \tag{2.5}$$

Using (2.5) in (2.4) and since $0 < \alpha \leq 1$, we obtain

$$x_n \geq (1 - p_n c^{\alpha-1}) z_n, \quad n \geq n_2. \tag{2.6}$$

From (2.6) and the fact that $\lim_{n\to\infty} p_n = 0$, for any $\delta \in (0,1)$, there is an integer $n_3 \geq n_2$ such that

$$x_n \geq \delta z_n \ \text{ for } \ n \geq n_3. \tag{2.7}$$

Fix $\delta \in (0,1)$. Since $n - \ell \to \infty$ as $n \to \infty$, one can choose an integer $n_4 \geq n_3$ such that $n - \ell \geq n_3$ for all $n \geq n_4$. Thus from (2.7) we have

$$x_{n-\ell} \geq \delta z_{n-\ell} \ \text{ for `} n \geq n_4. \tag{2.8}$$

Using (2.8) in equation (1.1) yields

$$\Delta^m z_n + \delta^{\beta} q_n z^{\beta}_{n-\ell} \leq 0 \tag{2.9}$$

or

$$\Delta^m z_n + \delta^\beta q_n z_{n-\ell}^{\beta-1} z_{n-\ell} \le 0, \quad n \ge n_4. \tag{2.10}$$

From (2.7) and $\beta > 1$, (2.10) yields

$$\Delta^M z_n + M q_n z_{n-\ell} \le 0, \quad n \ge n_4, \tag{2.11}$$

where $M = \delta^\beta c^{\beta-1}$. That is, (2.1) has a positive solution, which is a contradiction. This completes the proof. □

Theorem 2.5. *Let $\beta = 1$. If the even order linear delay difference inequality*

$$\Delta^m z_n + \delta q_n z_{n-\ell} \le 0 \tag{2.12}$$

has no positive solution for any $\delta \in (0,1)$, then equation (1.1) is oscillatory.

Proof. The proof follows from Theorem 2.4 with $\beta = 1$ and hence we omit the details. □

Next, we obtain an oscillation result for equation (1.1) when $0 < \beta < 1$.

Theorem 2.6. *Let $0 < \beta < 1$. If the even order linear delay difference inequality*

$$\Delta^m z_n + D\left((n-\ell)^{m-1}\right)^{\beta-1} q_n z_{n-\ell} \le 0 \tag{2.13}$$

has no positive solution for every constant $D > 0$, then equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1), say $x_n > 0$, $x_{n-k} > 0$ and $x_{n-\ell} > 0$ for all $n \ge n_1$ for some $n_1 \in \mathbb{N}(n_0)$. Arranging as in the proof of Theorem 2.4, we can arrive at (2.9) and this may be written as

$$\Delta^m z_n + \frac{\delta^\beta q_n}{z_{n-\ell}^{1-\beta}} z_{n-\ell} \le 0, \ n \ge n_4. \tag{2.14}$$

Since $\Delta^{m-1} z_n$ is positive and decreasing for all $n \ge n_3$, there is a constant $d > 0$ and an integer $n_5 \ge n_4$ such that

$$\Delta^{m-1} z_n \le d \ \text{ for all } \ n \ge n_5. \tag{2.15}$$

Summing (2.15) from n_5 to $n-1$ consecutively $m-1$ times, we deduce that

$$z_n \le M n^{m-1}, \quad n \ge n_4 \tag{2.16}$$

for some constant $M > 0$, and so

$$z_{m-\ell} \le M(n-\ell)^{m-1}, \quad n \ge n_5 \ge n_4. \tag{2.17}$$

Using (2.17) in (2.14) yields

$$\Delta^m z_n + \delta^\beta M^{\beta-1}\left((n-\ell)^{m-1}\right)^{\beta-1} q_n z_{n-\ell} \le 0$$

or

$$\Delta^m z_n + D\left((n-\ell)^{m-1}\right)^{\beta-1} q_n z_{n-\ell} \le 0, \quad n \ge n_5, \tag{2.18}$$

where $D = \delta^\beta M^{\beta-1} > 0$. The remainder of the proof is similar to that of Theorem 2.4 and hence is omitted. □

In the following theorems we obtain oscillation of equation (1.1) via comparison with first order delay difference equations whose oscillatory characters are known.

Theorem 2.7. *Let $\beta > 1$. If the first order linear delay difference equation*

$$\Delta w_n + M\frac{q_n}{(m-1)!}\left(\frac{n-\ell}{2^{m-1}}\right)^{m-1} w_{n-\ell} = 0 \tag{2.19}$$

is oscillatory for every constant $M > 0$, then equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1), say $x_n > 0$, $x_{n-k} > 0$ and $x_{n-\ell} > 0$ for all $n \geq n_1 \in \mathbb{N}(n_0)$. Proceeding as in the proof of Theorem 2.4, one obtain (2.11) for $n \geq n_3$. Since $z_n > 0$ and $\Delta z_n > 0$ for all $n \geq n_3$, we have $\lim_{n\to\infty} z_n \neq 0$, and so by Lemma 2.2, we have

$$z_n \geq \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1} \Delta^{m-1} z_n, \quad n \geq n_4 \geq 2^{m-1} n_1. \tag{2.20}$$

Using (2.20) in (2.11) yields

$$\Delta^m z_n + \frac{Mq_n}{(m-1)!}\left(\frac{n-\ell}{2^{m-1}}\right)^{m-1} \Delta^{m-1} z_{n-\ell} \leq 0, \quad n \geq n_5 \geq n_4. \tag{2.21}$$

With $w_n = \Delta^{m-1} z_n$, we see that $\{w_n\}$ is a positive solution of the first order delay difference inequality

$$\Delta w_n + \frac{Mq_n}{(m-1)!}\left(\frac{n-\ell}{2^{m-1}}\right)^{m-1} w_{n-\ell} \leq 0, \quad n \geq n_5. \tag{2.22}$$

Summing the inequality (2.22) from $n \geq n_5$ to u and letting $u \to \infty$, we obtain

$$w_n \geq \sum_{s=n}^{\infty} \frac{Mq_s}{(m-1)!}\left(\frac{s-\ell}{2^{m-1}}\right)^{m-1} w_{s-\ell}, \quad n \geq n_5.$$

The sequence $\{w_n\}$ is clearly decreasing for all $n \geq n_5$ and $\lim_{n\to\infty} z_n = 0$, which contradicts the fact that equation (2.19) is oscillatory. This completes the proof. □

Theorem 2.8. *Let $\beta = 1$. If the first order linear delay difference equation*

$$\Delta w_n + \frac{\delta}{(m-1)!} q_n \left(\frac{n-\ell}{2^{m-1}}\right)^{m-1} w_{n-\ell} = 0 \tag{2.23}$$

is oscillatory for every $\delta \in (0,1)$, then equation (1.1) is oscillatory.

Proof. The proof follows from (2.9) with $\beta = 1$, (2.20) and Theorem 2.7, and hence we omit the details. □

Lemma 2.9. *Let $\gamma \in (0,1]$ be ratio of odd positive integer. Assume $\{R_n\}$ is a positive real sequence for $n \in \mathbb{N}(n_0)$. If*

$$\lim_{n\to\infty} \sum_{s=n-\ell}^{n-1} R_s = \infty, \tag{2.24}$$

then the first order delay difference equation

$$\Delta y_n + R_n y_{n-\ell}^{\gamma} = 0 \tag{2.25}$$

is oscillatory.

Proof. Let $\{y_n\}$ be a positive solution of equation (2.25). One can observe that (2.24) implies that $\sum_{n=N}^{\infty} R_n = \infty$. Since $\{y_n\}$ is decreasing, there is a constant $M > 0$ such that $\lim_{n\to\infty} y_n = M \geq 0$. If $M > 0$, then summing (2.25) from N to n, we obtain

$$y_N \geq M^\alpha \sum_{s=N}^{n} R_s \to \infty \quad \text{as} \quad n \to \infty$$

which is a contradiction. Hence $\lim_{n\to\infty} y_n = 0$ and also $0 < y_n < 1$ and therefore $y_{n-\ell}^\alpha \geq y_{n-\ell}$. Using this in equation (2.25) we have

$$\Delta y_n + R_n y_{n-\ell} \leq 0, \quad n \geq N. \tag{2.26}$$

However the condition (2.24) and Theorem 7.6.1 of [8] imply that the inequality (2.26) has no positive solution, which is a contradiction. This completes the proof. □

By applying Lemma 2.9, we have the following results.

Corollary 2.10. *Let $\beta \geq 1$. If*

$$\lim_{n\to\infty} \sum_{s=n-\ell}^{n-1} q_s (s-\ell)^{m-1} = \infty \tag{2.27}$$

then equation (1.1) *is oscillatory.*

Proof. Applying Lemma 2.9, we see that equations (2.19) and (2.23) and so from Theorem 2.7 and 2.8, we conclude that equation (1.1) is oscillatory. This completes the proof. □

Theorem 2.11. *Let $0 < \beta < 1$. If the first order linear delay difference equation*

$$\Delta w_n + \frac{D}{(m-1)!} q_n \left((n-\ell)^{m-1}\right)^\beta w_{n-\ell} = 0 \tag{2.28}$$

is oscillatory for every constant $D > 0$, then equation (1.1) *is oscillatory.*

Proof. The proof follows from (2.18), (2.20) and Theorem 2.7 and hence the details are omitted. □

Corollary 2.12. *Let $0 < \beta < 1$. If*

$$\lim_{n\to\infty} \sum_{s=n-\ell}^{n-1} (s-\ell)^{\beta(m-1)} q_s = \infty \tag{2.29}$$

then equation (1.1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 2.11. □

3 Examples

In this section, we present two examples to illustrate the importance of the main results.

Example 3.1. *Consider the neutral difference equation*

$$\Delta^m \left(x_n + \frac{1}{n} x_{n-2}^{\frac{1}{3}} \right) + \frac{1}{n^\gamma} x_{n-3}^3 = 0, \quad n \geq 1. \tag{3.1}$$

where $m \geq 2$ *is even and* $\gamma > 0$. *Here* $p_n = \frac{1}{n}$, $q_n = \frac{1}{n^\gamma}$ *and* $\beta = 3$. *Then*

$$\sum_{s=n-3}^{n-1} (s-3)^{m-1} \frac{1}{s^\gamma} \approx \sum_{s=n-3}^{n-1} s^{m-\gamma-1} \approx 3n^{m-\gamma-1}$$

which $\to \infty$ *when* $m > \gamma + 1$. *Therefore by Corollary 2.10, every solution of equation* (3.1) *is oscillatory.*

Example 3.2. *Consider the neutral difference equation*

$$\Delta^m \left(x_n + \frac{1}{n} x_{n-2}^{\frac{1}{3}} \right) + \frac{1}{n^\gamma} x_{n-3}^{\frac{1}{3}} = 0, \quad n \geq 1. \tag{3.2}$$

where $m \geq 2$ *is even and* $\gamma > 0$. *Here* $p_n = \frac{1}{n}$, $q_n = \frac{1}{n^\gamma}$ *and* $\beta = \frac{1}{3}$. *Then*

$$\lim_{n\to\infty} \sum_{s=n-3}^{n-1} (s-3)^{\frac{1}{3}(m-1)} \approx 3n^{\frac{m-1}{3}-\gamma} \to \infty \quad as \quad n \to \infty$$

when $m > 3\gamma + 1$. *Therefore by Corollary 2.12, every solution of equation* (3.1) *is oscillatory.*

We conclude this paper with the following observations. There are many oscillation results available for first and higher order delay difference equations, and so it would be possible to formulate many criteria for the oscillation of equation (1.1) based on the results in this paper. Further the results presented in this paper are different from that of in [3], and hence our results are different and complement to the existing one.

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