Some new results on Locally Convex Tensor Products

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Introduction

In this paper we discuss about the locally convex Tensor products that E and f be vector spaces over the field K and define $^\wedge(E\text{x}F) = \{\sum(x,y)\alpha_{x,y} : \alpha_{x,y} = 0$ for all but a finite number of pairs (x,y)}.This is a vector space with operations

$$\sum_{x,y}(X,Y)\alpha_{x,y} + \sum_{x,y}(X,Y)\beta_{x,y} = \sum_{x,y}(X,Y)(\alpha_{x,y} + \beta_{x,y})$$

$$\beta\sum_{x,y}(X,Y)\alpha_{x,y} = \sum_{x,y}(X,y)\beta.\alpha_{x,y}$$

Now we define a subspace of $^\wedge(E\text{x}F)$ by

$^{\wedge 0}(E\text{x}F)$= linear hull of all elements of the form

$(\sum_{i=1}^{n} X_i\,\alpha_i, \sum_{k=1}^{m} y_k\beta_k = \sum_{i=1}^{n}\sum_{k=1}^{m}(X_iY_k)\alpha_i\beta_k$. **.........................(1)**

Keywords

Tensor products, linearly independent, projective topology, absolutely convex

Definition (1)

The tensor products E⊗F is defined o be the space $\frac{\wedge}{\wedge_0}$

Let X: ^ → $\frac{\wedge}{\wedge_0}$ be the canonical map. So , X(^(ExF)) =E⊗F

We denote the element X(x,y) in E⊗F by X⊗Y . The following rule of calculation in E⊗F follows from (1)

$(\sum_{i=1}^{n} X_i\,\alpha_i) \otimes (\sum_{k=1}^{m} y_k\beta_k) = \sum_{i=1}^{n}\sum_{k=1}^{m}(X_i \otimes Y_k)\alpha_i\beta_k$ **.................................(2)**

According , 0 ⊗ Y=(o) ⊗Y= (0 ⊗Y)0 = 0 , likewise X ⊗ 0 =0

Proposition (1)

The tensor product is commutative i.e E⊗ F is isomorphic to F⊗ E as a vector space under the correspondence X ⊗ Y→Y ⊗ X

It follows immediately from the equation (X⊗ Y)α = X ⊗ (Yα) that every element of E ⊗ F can be written in the form $\sum_{i=1}^{n}(X_i ⊗ Y_i)$.

Proposition (2)

If y_1, y_2, y_3, y_4.............................y_n are linearly independent elements of F ,it follows from

$\sum_{i=1}^{n}(X_i⊗Y_i)$= 0 that X_i= 0 for i= 1,2,.............,n.

Proposition (3)

If $\{X_v\}$,$v \in N$, is a basis for E and $\{y_\mu\}$, $\mu \in M$, is a basis for F ,then $\{X_v⊗Y_\mu\}$, (v, μ) ∈N x M

Is a basis for E x F . If E has dimension d and F has dimension e, then E ⊗ F has dimension.

It follows from the above proposition that if A and B are linear subspace of E and F respectively, then A ⊗ B is isomorphic to the linear subspace of E ⊗ F spanned by the elements a⊗ b, a∈A ,b∈B.

Proposition (4)

Every element $Z \neq 0$ of E ⊗ F has a representation $Z = \sum_{i=1}^{r} X^{(i)}⊗Y^{(i)}$ for which both $X^{(i)}$and $Y^{(i)}$ are linearly independent .

Let E and F be locally convex vector spaces and let A and B be subsets of E and F respectively. We define the set

A⊗ B={a ⊗ b: a∈A ,b∈B}

We note that this definition introduces a certain abuse of notation .The tensor product E ⊗ F of vector space exhausted by elements of the form X ⊗ Y , we must take finite sums $\sum X_i ⊗Y_i$.

Let U and v be closed absolutely convex nhds. of) in E and F respectively . From the set ɼ(U⊗V) = absolutely convex hull of U ⊗ V in E ⊗ V .

Proposition (5)

Let P(X) and q(y) be the semi – norms defines by U and V respectively. The set ɼ(U ⊗ V) is absorbing and thus define a semi-norms . The semi-norm of ɼ(U ⊗ V) is given by

P ⊗ q(Z) = inf $\sum_{i=1}^{n} p(x_i)q(y_i)$ ....(1)

Where the inf. Is taken over all representation $z = \sum x_i⊗y_i$ in E ⊗ F.

Proposition (6)

$E' ⊗ F' \sqsubseteq$(E ⊗πF)$'$

And < $E' ⊗ F'$, E ⊗πF> forms a dual system.

Proposition (7)

If p and q are semi-norms on E and F respectively ,then for elements of the form x ⊗ y in E ⊗ F ,

P ⊗ q(x ⊗ y)= P(x) q(y)

Proposition (8)

The projective tensor product E ⊗πF of two normed spaces E, p an f , q is normed space with norm p ⊗ q.

If E and F ae metrizable locally convex spaces with semi norms $p_1 \leq p_2$.......................and $q_1 \leq q_2$......................................respectively, then, E ⊗$_\pi$ F is metrizable with defining semi- norms

$p_1 ⊗ q_1 \leq p_2 ⊗ q_2 \leq$ ....

Theorem (1)

The projective topology π is the finest locally convex topology on E ⊗ F for which the canonical map

$$X: E \; x \; F \rightarrow E ⊗ F$$

Definition (2)

We denote the completion of E ⊗$_\pi$F by E $\tilde{x}$ F

Proposition (9)

E x F is algebraically isomorphic to β (E' x F') where E' has the weak * $T_s(E)$, topology and F' has any topology between the weak * ,$T_s(E)$ and the Mackey ,$T_k(F)$, topologies.

Proposition (10)

If E and F are Frechet spaces ,then E ⊗$_\pi$F is barrelled.

Proof

We show that every weakly bounded set of (E ⊗$_\pi$F)$'$ is equicontinuous . Let $\bar{M}$= {B} be a weakly bounded subset of (E ⊗$_\pi$F)$'$ and let M= B be the corresponding subset of B (E x F) . Now $\bar{M}$ weakly bounded means given X x Y ,there is a constant K with

|$\bar{B}(x$ x y)| ≤ K for all $\bar{B}$ ∈ M

Then also | B(x,y)|≤ K ,for all B∈ M

So, M is weakly bounded in B(E x F) .then we know by a theorem on continuity that M is equi continuous. Again M is then an equi continuous subset of (E ⊗$_\pi$F)$'$. Suppose M_1 and M_2 are bounded in the locally convex spaces E and F respectively. Then ɽ(M_1x M_2) is bounded in E ⊗$_\pi$F it is natural to ask if all bounded sets in E ⊗$_\pi$F arise in this way from bounded sets in E and F.

We know that the bibounded topology on B(E x F) is defined by nhds, of O, u_{M_1M2}, where M_1 and M_2 are bounded in E and F respectively, and

$u_{M_1M_2} = \{B \in B(E$ x F): | B($M_1, M_2)| \leq 1\}$

$|B(M_1, M_2)| \leq 1$ **iff** $|\widehat{B}(M_1 \text{x } M_2)| \leq 1$

Iff $|\widehat{B}(M_1 \text{x } M_2)| \leq 1$ **iff** $\widehat{B} \in \Gamma(\overline{M_1 x\, M_2})^0$

We also know that the strong topology on $(E \otimes_n F)'$ is defined by poars of bounded sets in $(E \otimes_n F)$.

Therfore the questions posed bove is equivalent to the question : Is the strong topology on

$(E \otimes_n F)'$ identical with the bibounded topology on B (E x F) .

For banach space ,the strong topology T_b ($(E \otimes_n F)$ on $(E \otimes_n F)'$ is identical with the bibounded topology on B(E xF) .This follows from the fact that $\Gamma(U$ x V) is the unit ball of $(E \otimes_n F)$ if U and V are the unit balls of E and F respectively.

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