ON \((1, 2)^*\omega\)-CLOSED SETS

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Abstract. In this paper is to introduce and study the concepts of \((1, 2)^*\omega\)-closed sets in bitopological spaces. The relations between \((1, 2)^*\omega\)-closed sets with various closed sets are analyzed. Finally, some more properties are investigate on line of research.

1. Introduction

In 1963, J.C.Kelly [3] expressed the geometrical existence of bitopological space that is a non empty set \(X\) together with two arbitrary topologies defined on \(X\) and it plays an important role to study the shapes of objects. General topologist have introduced and investigated different forms of open sets in bitopological space. N. Levine [6] introduced generalized closed (briefly g-closed) sets and studied their basic properties. In 1992, J. Fukutake [1] further studied the bitopological concept and defined the Baire space. In 1991, M. Lellis Thivagar \textit{et al.}, [4] established the properties of new type of bitopological open sets which are entirely different from Kellys pairwise open sets called \(\tau_{12}\)-open set and \(\tau_{1}\tau_{2}\)-open set. In 2006, M. Lellis

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Thivagar et al., [5] also discussed the behaviour of \((1, 2)^*\)-open set, \((1, 2)^*\)-semi open set with their continuity and defined various types of bitopological generalized closed sets such as \((1, 2)^*\)-sg-closed, \((1, 2)^*\)-gs-closed sets and so on.

In this paper is to introduce and study the concepts of \((1, 2)^*\)-\(\psi\omega\)-closed sets in bitopological spaces. The relations between \((1, 2)^*\)-\(\psi\omega\)-closed sets with various closed sets are analyzed. Finally, some more properties are investigate on line of research.

2. Preliminaries

Throughout this paper, \((X, \tau_1, \tau_2)\) (briefly, \(X\)) will denote bitopological spaces.

**Definition 2.1.** Let \(S\) be a subset of \(X\). Then \(S\) is said to be \(\tau_1, \tau_2\)-open [8] if \(S = A \cup B\) where \(A \in \tau_1\) and \(B \in \tau_2\).

The complement of \(\tau_1, \tau_2\)-open set is called \(\tau_1, \tau_2\)-closed.

Notice that \(\tau_1, \tau_2\)-open sets need not necessarily form a topology.

**Definition 2.2.** [8] Let \(S\) be a subset of a bitopological space \(X\). Then

(1) the \(\tau_1, \tau_2\)-closure of \(S\), denoted by \(\tau_1, \tau_2\text{-cl}(S)\), is defined as \(\cap \{F : S \subseteq F\text{ and } F \text{ is } \tau_1, \tau_2\text{-closed}\}\).

(2) the \(\tau_1, \tau_2\)-interior of \(S\), denoted by \(\tau_1, \tau_2\text{-int}(S)\), is defined as \(\cup \{F : F \subseteq S\text{ and } F \text{ is } \tau_1, \tau_2\text{-open}\}\).

**Definition 2.3.** [8] A subset \(A\) of a bitopological space \(X\) is called \(\tau_1, \tau_2\)-semi-open set if \(A \subseteq \tau_1, \tau_2\text{-cl}(\tau_1, \tau_2\text{-int}(A))\). The complement of open set is said to be closed.

**Definition 2.4.** A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called

(1) \((1, 2)^*\)-g-closed set [9] if \(\tau_1, \tau_2\text{-cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_1, \tau_2\)-open.

(2) \((1, 2)^*\)-gs-closed set [7] if \((1, 2)^*\text{-scl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_1, \tau_2\)-open.
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(3) \((1,2)^\ast\)-\(\hat{g}\)-closed set \(= (1,2)^\ast\)-\(\omega\)-closed \([2]\) if \(\tau_{1,2}\)-cl\(A\) \(\subseteq U\) whenever \(A \subseteq U\) and \(U\) is \((1,2)^\ast\)-semi open.

The complements of the above mentioned closed sets are called their respective open sets.

3. ON \((1,2)^\ast\)-\(\psi\omega\)-CLOSED SETS

**Definition 3.1.** A subset \(H\) of a bitopological space \((X,\tau_1,\tau_2)\) is called a \((1,2)^\ast\)-\(\psi\omega\)-closed set if \(H \subseteq K, K \in (1,2)^\ast\)-\(\omega\)-open \(\Rightarrow (1,2)^\ast\)-\(\psi\)cl\((H) \subseteq K\). The complement of \((1,2)^\ast\)-\(\psi\omega\)-closed set is said to be a \((1,2)^\ast\)-\(\psi\omega\)-open set.

**Theorem 3.2.** In a bitopological space \((X,\tau_1,\tau_2)\), every \((1,2)^\ast\)-\(\omega\)-closed set is \((1,2)^\ast\)-\(\psi\omega\)-closed.

**Proof.** \(H\) is a \((1,2)^\ast\)-closed set of \((X,\tau_1,\tau_2)\). Let \(K\) be a \((1,2)^\ast\)-\(\omega\)-open set, such that \(H \subseteq K\). Since \(H\) is closed, then \((1,2)^\ast\)-cl\((H) \subseteq \tau_{1,2}\)-cl\((H) = H \subseteq K\). Hence \(H\) is \((1,2)^\ast\)-\(\psi\omega\)-closed.

**Remark 3.3.** The converse of Theorem 3.2 is need not be true in general as shown in the following Example.

**Example 3.4.** Let \(X = \{p_1, p_2, p_3\}\) be a set with the topology \(\tau_1 = \{\phi, X, \{p_1\}, \{p_1, p_2\}\}\) and the topology \(\tau_2 = \{\phi, X\}\) then the bitopology \(\tau_{1,2} = \{\phi, X, \{p_1\}, \{p_1, p_2\}\}\). In the bitopological space \((X,\tau_1,\tau_2)\), then the subset \(\{p_1, p_3\}\) is \((1,2)^\ast\)-\(\psi\omega\)-closed but not \((1,2)^\ast\)-\(\omega\)-closed.

**Theorem 3.5.** In a bitopological space \((X,\tau_1,\tau_2)\), every \((1,2)^\ast\)-semi closed is \((1,2)^\ast\)-\(\psi\omega\)-closed set.

**Proof.** \(H\) is a \((1,2)^\ast\)-semi closed set of \((X,\tau_1,\tau_2)\). Let \(K\) be a \((1,2)^\ast\)-\(\omega\)-open set of \((X,\tau_1,\tau_2)\) such that \(H \subseteq K\). Since \(H\) is \((1,2)^\ast\)-semi closed, then \((1,2)^\ast\)-scl\((H) = H\).
For each subset $H$ of $X$, therefore $(1,2)^*\psi cl(H) \subseteq (1,2)^*-scl(H) = H \subseteq K$ and so $(1,2)^*\psi cl(H) \subseteq K$. Hence $H$ is $(1,2)^*\psi \omega$-closed.

**Remark 3.6.** The converse of Theorem 3.5 is need not be true in general as shown in the following Example.

**Example 3.7.** Let $X = \{1,2,3\}$ be set with the topology $\tau_1 = \{\phi, X, \{1\}, \{1,2\}\}$ and $\tau_2 = \{\phi, X\}$ then the bitopology $\tau_{1,2} = \{\phi, X, \{1\}, \{1,2\}\}$. In the bitopological space $(X, \tau_1, \tau_2)$, then the subset $\{1,3\}$ is $(1,2)^*\psi \omega$-closed but not $(1,2)^*$-semi closed.

**Theorem 3.8.** In a bitopological space $(X, \tau_1, \tau_2)$, every $(1,2)^*\psi \omega$-closed set is $(1,2)^*$-g-closed.

**Proof.** Follows from the Definition.

**Remark 3.9.** The converse of Theorem 3.8 is need not be true in general as shown in the following Example.

**Example 3.10.** In Example 3.7, then the subset $\{1,2\}$ is $(1,2)^*$-g-closed but not $(1,2)^*\psi \omega$-closed.

**Theorem 3.11.** In a bitopological space $(X, \tau_1, \tau_2)$, every $(1,2)^*\psi \omega$-closed set is $(1,2)^*$-gs-closed.

**Proof.** $H$ is a $(1,2)^*\psi \omega$-closed and $K$ be any $(1,2)^*$-open set containing $H$ in $(X, \tau_1, \tau_2)$. Since every $(1,2)^*$-open set is $(1,2)^*\omega$-open, $(1,2)^*\psi cl(H) \subseteq K$ for every subset $H$ of $(X, \tau_1, \tau_2)$. Since $(1,2)^*scl(H) \subseteq (1,2)^*\psi cl(H) \subseteq K$, $(1,2)^*scl(H) \subseteq K$ and hence $H$ is $(1,2)^*$-gs-closed.

**Remark 3.12.** The converse of Theorem 3.11 is need not be true in general as shown in the following Example.
Example 3.13. Let $X = \{a, b, c\}$ be a set with the topology $\tau_1 = \{\emptyset, X, \{a\}, \{c\}, \{b, c\}\}$ and $\tau_2 = \{\emptyset, X\}$ then the bitopology $\tau_{1,2} = \{\emptyset, X, \{a\}, \{c\}, \{b, c\}\}$. In the bitopological space $(X, \tau_1, \tau_2)$, then the subset $\{c\}$ is $(1, 2)^*-\text{gs}-\text{closed}$ but not $(1, 2)^*-\psi\omega$-closed.

Theorem 3.14. In a bitopological space $(X, \tau_1, \tau_2)$, every $(1, 2)^*-\psi\omega$-closed set is $(1, 2)^*-\psi g$-closed.

Proof. $H$ is a $(1, 2)^*-\psi\omega$-closed and $K$ is each $(1, 2)^*-\text{open}$ set containing $H$. Since every $(1, 2)^*-\text{open}$ set is $(1, 2)^*-\psi g$-open, $(1, 2)^*-\text{cl}(H) \subseteq K$, whenever $H \subseteq K$ and $K$ is $(1, 2)^*-\psi g$-open. Therefore $(1, 2)^*-\text{cl}(H) \subseteq K$ and $K$ is $(1, 2)^*-\text{open}$. Thus $H$ is $(1, 2)^*-\psi\omega$-closed.

Remark 3.15. The converse of Theorem 3.14 is need not be true true in general as shown in the following Example.

Example 3.16. Let $X = \{x, y, z\}$ be a set with the topology $\tau_1 = \{\emptyset, X, \{y\}\}$ and $\tau_2 = \{\emptyset, X\}$ then the bitopology $\tau_{1,2} = \{\emptyset, X, \{y\}\}$. In the bitopological space $(X, \tau_1, \tau_2)$. Then the subset $\{y\}$ is $\psi g$-closed but not $\psi\omega$-closed.

Theorem 3.17. In a bitopological space $(X, \tau_1, \tau_2)$, union of two $(1, 2)^*-\psi\omega$-closed is $(1, 2)^*-\psi\omega$-closed.

Proof. Suppose that $H$ and $K$ are $(1, 2)^*-\psi\omega$-closed sets in $(X, \tau_1, \tau_2)$. Let $K$ be $(1, 2)^*-\psi g$-open in $X$ such that $H \cup K \subseteq G$. Then $H \subseteq G$ and $K \subseteq G$. Since $H$ and $K$ are $(1, 2)^*-\psi\omega$-closed, $(1, 2)^*-\text{cl}(H) \subseteq G$ and $(1, 2)^*-\text{cl}(K) \subseteq G$. Hence $(1, 2)^*-\text{cl}(H \cup K) = (1, 2)^*-\text{cl}(H) \cup (1, 2)^*-\text{cl}(K) \subseteq G$. That is $(1, 2)^*-\text{cl}(H \cup K) \subseteq G$. Therefore $H \cup K$ is $(1, 2)^*-\psi\omega$-closed set.
4. Some more properties

**Theorem 4.1.** Let $H$ be a $(1,2)^{*}$-$\psi\omega$-closed set of a bitopological space $(X, \tau_1, \tau_2)$. Then $\psi cl(H) - H$ does not contain a non-empty $(1,2)^{*}$-$\omega$-closed set.

*Proof.* Assuming that $H$ is $(1,2)^{*}$-$\psi\omega$-closed set, let $A$ be a $(1,2)^{*}$-$\omega$-closed set contained in $(1,2)^{*}$-$\psi cl(H) - H$. Now $A^c$ is $(1,2)^{*}$-$\psi\omega$-open set of $(X, \tau_1, \tau_2)$ such that $H \subseteq A^c$. Since $H$ is $(1,2)^{*}$-$\psi\omega$-closed set of $(X, \tau_1, \tau_2)$, then $(1,2)^{*}$-$\psi cl(H) \subseteq A^c$. Thus $A \subseteq ((1,2)^{*}$-$\psi cl(H))^c$. Also $A \subseteq (1,2)^{*}$-$\psi cl(H) - H$. Therefore $A \subseteq ((1,2)^{*}$-$\psi cl(H))^c \cap ((1,2)^{*}$-$\psi cl(H)) = \phi$. Hence $A = \phi$.

**Theorem 4.2.** If $H$ is $(1,2)^{*}$-$\psi\omega$-closed set in $(X, \tau_1, \tau_2)$ and $H \subseteq K \subseteq (1,2)^{*}$-$\psi cl(H)$. Then $K$ is $(1,2)^{*}$-$\psi\omega$-closed set.

*Proof.* Let $G$ be a $(1,2)^{*}$-$\omega$-open set of $(X, \tau_1, \tau_2)$ such that $K \subseteq G$. Then $H \subseteq G$. Since $H$ is $(1,2)^{*}$-$\psi\omega$-closed set, $(1,2)^{*}$-$\psi cl(H) \subseteq G$. Also since $K \subseteq (1,2)^{*}$-$\psi cl(H)$, $(1,2)^{*}$-$\psi cl(K) \subseteq (1,2)^{*}$-$\psi cl((1,2)^{*}$-$\psi cl(H)) = (1,2)^{*}$-$\psi cl(H)$. Hence $(1,2)^{*}$-$\omega cl(K) \subseteq G$. Therefore $K$ is also a $(1,2)^{*}$-$\omega$-closed set.

**Theorem 4.3.** For each $a \subseteq X$ either $[a]$ is $(1,2)^{*}$-$\omega$-closed or $[a]^c$ is $(1,2)^{*}$-$\psi\omega$-closed set.

*Proof.* Assuming that $[a]$ is not $(1,2)^{*}$-$\omega$-closed in $X$. Then only $[a]^c$ is not $(1,2)^{*}$-$\omega$-open and the only $(1,2)^{*}$-$\omega$-open set containing $[a]^c$ is the space $X$ itself. That is $[a]^c \subseteq X$. Therefore $(1,2)^{*}$-$\psi cl(H)([a]^c) \subseteq X$ and so $[a]^c$ is $(1,2)^{*}$-$\psi\omega$-closed set.

**Theorem 4.4.** Let $q$ be a $(1,2)^{*}$-$\psi\omega$-closed set in $X$. Then $H$ is $(1,2)^{*}$-$\omega$-closed set if and only if $(1,2)^{*}$-$\psi cl(H) - H$ is closed.

*Proof.* $\Rightarrow$ If $H$ is a $(1,2)^{*}$-$\psi\omega$-closed subset of $X$. Then $(1,2)^{*}$-$\psi cl(H) = H$ and so $(1,2)^{*}$-$\psi cl(H) - H = \phi$ which is $(1,2)^{*}$-closed.
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\[ \Leftarrow \text{Since } H \text{ is } (1,2)^*\-\psi_\omega\text{-closed set by Theorem 4.1, } (1,2)^*\-\psi cl(H) = H \text{ contains no nonempty } (1,2)^*\text{-closed set. But } (1,2)^*\-\psi cl(H) = H \text{ is } (1,2)^*\text{-closed. This implies } (1,2)^*\-\psi cl(H) = H = \emptyset. \text{ That is } (1,2)^*\-\psi cl(H) = H. \text{ Hence } H \text{ is } (1,2)^*\-\omega\text{-closed.} \]

**Definition 4.5.** The intersection of all \((1,2)^*\-\omega\)-open subsets of a bitopological space \((X, \tau_1, \tau_2)\) containing \(H\) is called the \((1,2)^*\-\omega\)-kernel of \(H\) and is denoted by \((1,2)^*\-\omega\text{-ker}(H)\).

**Theorem 4.6.** A subset \(H\) of a bitopological space \((X, \tau_1, \tau_2)\) is \((1,2)^*\-\psi_\omega\text{-closed set }\) \(\iff (1,2)^*\-\psi cl(H) \subseteq (1,2)^*\-\omega\text{-ker}(H).\)

*Proof.* Suppose that \(H\) is \((1,2)^*\-\psi_\omega\text{-closed set in } X\). Then \((1,2)^*\-\psi cl(H) \subseteq G\) whenever \(H \subseteq G\) and \(G\) is \((1,2)^*\-\omega\text{-open. Let } a \in (1,2)^*\-\psi cl(H). \text{ If } a \notin (1,2)^*\-\omega\text{-ker}(H),\) then there is a \((1,2)^*\-\omega\text{-open set } G\) such that \(a \notin G. \text{ Since } G\) is a \((1,2)^*\-\omega\text{-open set containing } H, \text{ we have } a \notin (1,2)^*\-\psi cl(H),\) which is a contradiction.

Conversely, let \((1,2)^*\-\psi cl(H) \subseteq (1,2)^*\-\omega\text{-ker}(H).\) If \(a\) is any \((1,2)^*\-\omega\text{-open set containing } H, \text{ then } (1,2)^*\-\psi cl(H) \subseteq (1,2)^*\-\omega\text{-ker}(H) \subseteq G. \text{ Therefore } H\) is \((1,2)^*\-\psi_\omega\text{-closed set.} \)

**Definition 4.7.** A space \((X, \tau_1, \tau_2)\) is called \((1,2)^*\-\tau_\psi_\omega\text{-space, if every } (1,2)^*\-\psi_\omega\text{-closed set in it is } (1,2)^*\-\omega\text{-closed.}\)

**Theorem 4.8.** For a bitopological space \((X, \tau_1, \tau_2)\), the following relations are equivalent.

1. \((X, \tau_1, \tau_2)\) is a \((1,2)^*\-\tau_\psi_\omega\text{-space.}\)
2. Every singleton \([a]\) is either \((1,2)^*\-\omega\text{-closed or } (1,2)^*\-\psi\text{-open.}\)

*Proof.* (1) \(\Rightarrow\) (2) Let \(a \subseteq X. \text{ Suppose } [a]\) is not a \((1,2)^*\-\omega\text{-closed set of } (X, \tau_1, \tau_2).\) Then \(X - [a]\) is not a \((1,2)^*\-\omega\text{-open set. Thus } X - [a]\) is an \((1,2)^*\-\psi_\omega\text{-closed set of} \)
(X, τ₁, τ₂). Since (X, τ₁, τ₂) is (1, 2)*-τψω-closed, X − {a} is a (1, 2)*-ψ-closed set of (X, τ₁, τ₂). i.e., {a} is (1, 2)*-ψ-open set of (X, τ₁, τ₂).

(2) ⇒ (1) If H is a (1, 2)*-ψω-closed set of (X, τ₁, τ₂). Let x ∈ (1, 2)*-ψcl(H). By (2) [x] is either (1, 2)*-ω-closed or (1, 2)*-ψ-open.

Case-A: If [x] is a (1, 2)*-ω-closed set of (X, τ₁, τ₂). Assuming that x /∈ H, then we have x ∈ (1, 2)*-ψcl(H) − H. Hence x ∈ H.

Case-B: If [x] is a (1, 2)*-ψ-open set of (X, τ₁, τ₂). Since X ∈ (1, 2)*-ψcl(H), then [x] ∩ H = φ. This shows that x ∈ H. So in both the cases we have H ⊆ (1, 2)*-ψcl(H). Therefore H = (1, 2)*-ψcl(H) or equivalently H is (1, 2)*-closed. Hence (X, τ₁, τ₂) is a (1, 2)*-τψω-space.

**Theorem 4.9.** In a bitopological space (X, τ₁, τ₂), let H be a subset of (1, 2)*-ω-open set and (1, 2)*-ψω-closed ⇒ H is a subset of (1, 2)*-ω-closed.

**Proof.** Since H is (1, 2)*-ω-open and (1, 2)*-ψω-closed set, (1, 2)*-ψcl(H) ⊆ H. Hence H is (1, 2)*-ψ-closed.

**References**


