SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC AND CONJUGATE POINTS DEFINED USING q-DIFFERENTIAL OPERATOR

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Abstract. In this paper, we define a class of analytic functions using q-differential operator which are related to the classes of starlike functions with respect to symmetric points and convex functions with respect to symmetric points. We obtain the inequalities for the Maclaurin-Taylor coefficients of the functions belonging to the defined subclasses.

1. INTRODUCTION

We start with a very brief introduction on q-calculus and the notations which are required for our study. Quantum calculus popularly called as q-calculus is based on the idea of finite difference re-scaling. The difference of quantum differentials from the ordinary ones is that notion of limit is removed in q-calculus, that is q-derivative is merely a ratio which is given by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}.$$

Notice that as limit $q \to 1^-$, $D_q f(z) = f'(z)$. q-calculus has numerous applications in variety of disciplines such as theory of special functions, operator theory, quantum-mechanics, relativity etc. Notations and symbols play an very important role in the study of q-calculus. Throughout this paper, we let

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \quad [0]_q = 0, \quad (q \in \mathbb{C}).$$

Let $\mathcal{U}$ be the class of functions which are analytic and univalent in the open unit disc $D = \{z : |z| < 1\}$ given by

$$w(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

and satisfying the conditions

$$w(0) = 0, \, |w(z)| < 1, \, z \in D.$$

Let $\mathcal{S}$ denote the class of functions f which are analytic and univalent in D of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \, z \in D. \tag{1.1}$$

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We let $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ to denote the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike of order β and convex of order β in $\mathcal{U}$. The class $\mathcal{P}$ denote the class of functions of function of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ that are analytic in $\mathcal{U}$ and such that $Re\,(p(z)) > 0$ for all z in $\mathcal{U}$. For detailed study of various subclasses of univalent function theory, we refer to [4, 6].

Let $f(z)$ and $g(z)$ be analytic in D. Then we say that the function $f(z)$ is subordinate to $g(z)$ in D, if there exists an Schwartz function $w(z)$ in D such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in D, then the subordination is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$.

Using the concept of subordination of analytic functions, Ma and Minda[10] introduced the class $\mathcal{S}^*(\phi)$ of functions in $\mathcal{A}$ satisfying $\frac{zf'(z)}{f(z)} \prec \phi$ where $\phi \in \mathcal{P}$ with $\phi'(0) > 0$ maps $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to real axis. This class specializes to several classes of univalent functions for suitable choice of ϕ. For instance, the class $\mathcal{S}^*(\frac{1+Az}{1+Bz}) =: \mathcal{S}^*[A;B]$ where $-1 \le B < A \le 1$, is the class of the Janowski starlike functions(see [7]).

Let S_s^* be the subclass of S consisting of functions given by (1.1) satisfying

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0,\ z \in D$$

Those functions in S_s^* are called starlike with respect to symmetric points and were introduced by Sakaguchi [12] in 1959. Ashwah and Thomas in [3] introduced another class namely the class S_c^* consisting of functions starlike with respect to conjugate points. Let S_c^* be the subclass of $\mathcal{S}$ consisting of functions given by (1.2) and satisfying the condition

$$Re\left\{\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}}\right\} > 0,\ z \in D$$

Motivated by S_s^* ,many authors discussed the following class C_s of function convex with respect to symmetric points and its subclasses.

Let C_s be the subclass of $\mathcal{S}$ consisting of functions given by (1.1) and satisfying the condition

$$Re\left\{\frac{zf'(z)}{(f(z) - f(-z))'}\right\} > 0,\ z \in D$$

In terms of subordination, Goel and Mehrok in 1982 introduced a subclass of S_s^* denoted by $S_s^*(A,B)$. Let $S_s^*(A,B)$ be the class of functions of the form(1.1) and satisfying the condition

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1+Az}{1+Bz},\ -1 \le B < A \le 1, z \in D.$$

Also let $S_c^*(A,B)$ be the class of functions of the form (1.1) and satisfying the condition

$$\frac{2zf'(z)}{\left(f(z) + \overline{f(\overline{z})}\right)} \prec \frac{1+Az}{1+Bz},\ -1 \le B < A \le 1, z \in D.$$

Let $C_s(A,B)$ be the class of functions of the form(1.1) and satisfying the condition

$$\frac{2\left(zf'(z)\right)'}{\left(f(z)-f(-z)\right)'} \prec \frac{1+Az}{1+Bz}, \; -1 \leq B < A \leq 1, z \in D.$$

Also let $C_c(A,B)$ be the class of functions of the form(1.1) and satisfying the condition

$$\frac{2\left(zf'(z)\right)'}{\left(f(z)+\overline{f(\overline{z})}\right)'} \prec \frac{1+Az}{1+Bz}, \; -1 \leq B < A \leq 1, z \in D.$$

In this paper, we introduce the class $M_s^q(\alpha, A, B)$ consisting of analytic functions f of the form (1.1) and satisfying

$$\frac{2zD_q f(z) + 2\alpha z^2 D_q^2 f(z)}{(1-\alpha)\left(f(z)-f(-z)\right) + \alpha z D_q \left(f(z)-f(-z)\right)} \prec \frac{1+Az}{1+Bz}$$

$$-1 \leq B < A \leq 1, \; 0 \leq \alpha \leq 1, \; z \in D.$$

We note that if $q \to 1^-$, $M_s^q(0, A, B) = S_s^*(A, B)$ and $M_s^q(1, A, B) = C_s(A, B)$. Also we introduce the class $M_c(\alpha, A, B)$ consisting of analytic functions f of the form (1.1) and satisfying

$$\frac{2zD_q f(z) + 2\alpha z^2 D_q^2 f(z)}{(1-\alpha)\left(f(z)+\overline{f(\overline{z})}\right) + \alpha z D_q \left(f(z)+\overline{f(\overline{z})}\right)} \prec \frac{1+Az}{1+Bz}$$

$$-1 \leq B < A \leq 1, \; 0 \leq \alpha \leq 1, \; z \in D.$$

Note that if $q \to 1^-$, we have $M_c^q(0, A, B) = S_c^*(A, B)$ and $M_c^q(1, A, B) = C_c(A, B)$.

By definition of subordination, it follows that $f \in M_c^q(\alpha, A, B)$ if and only if

$$\text{(1.2)} \quad \frac{2zD_q f(z) + 2\alpha z^2 D_q^2 f(z)}{(1-\alpha)\left(f(z)-f(-z)\right) + \alpha z D_q \left(f(z)-f(-z)\right)} = \frac{1+A\,w(z)}{1+B\,w(z)} = p(z), \; w \in \mathcal{U}$$

and that $f \in M_c^q(\alpha, A, B)$ if and only if

$$\text{(1.3)} \quad \frac{2zD_q f(z) + 2\alpha z^2 D_q^2 f(z)}{(1-\alpha)\left(f(z)+\overline{f(\overline{z})}\right) + \alpha z D_q \left(f(z)+\overline{f(\overline{z})}\right)} = \frac{1+A\,w(z)}{1+B\,w(z)} = p(z), \; w \in \mathcal{U}$$

where

$$\text{(1.4)} \quad p(z) = 1 + \sum_{n=2}^{\infty} P_n z^n.$$

We study the classes $M_s^q(\alpha, A, B)$ and $M_c^q(\alpha, A, B)$, the coefficient estimates for functions belonging to these classes are obtained.

2. Preliminary result

We need the following lemma for proving our results.

Lemma 2.1. [11] *let* $f(z)=\sum_{n=1}^{\infty}a_nz^n$ *be analytic in* $\mathcal{U}$ *and* $g(z)=\sum_{n=1}^{\infty}b_zz^n$ *is analytic and convex in* $\mathcal{U}$. *If* $f(z)\prec g(z)$, *then* $|a_n|\le|b_1|$, *for* $n=1,\,2,\,\ldots$.

As application of the Lemma 2.1, we have the following result.

Lemma 2.2. [5]
Let $p(z)$ *be given by (1.4) satisfy*

$$p(z)=\frac{1+A\,w\,(z)}{1+B\,w\,(z)},\qquad (w(z)\in\mathcal{U}),$$

then

$$|\,P_n\,|\le A-B,\quad n\,=\,1\,,2\,,3\,\ldots. \tag{2.1}$$

3. Main result

We give the coefficient inequalities for the classes $M_s^q(\alpha,A,B)$ and $M_c^q(\alpha,A,B)$.

Theorem 3.1. *Let* $f\in M_s^q(\alpha,A,B)$. *Then for* $n\ge1,0\le\alpha\le1$,

$$|a_{2n}|\le\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)^n\left(\frac{q^n-1}{q-1}\right)!\left(1+\alpha q\left(\frac{q^{2n-1}-1}{q-1}\right)\right)}\prod_{j=1}^{n-1}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right) \tag{3.1}$$

$$|a_{2n+1}|\le\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)^n\left(\frac{q^n-1}{q-1}\right)!\left(1+\alpha q\left(\frac{q^{2n}-1}{q-1}\right)\right)}\prod_{j=1}^{n-1}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right). \tag{3.2}$$

Proof. From(1.1) and (1.4), we have

$$\left[z+\left(\frac{q^2-1}{q-1}\right)a_2z^2+\left(\frac{q^3-1}{q-1}\right)a_3z^3+\left(\frac{q^4-1}{q-1}\right)a_4z^4+\left(\frac{q^5-1}{q-1}\right)a_5z^5+\cdots+\left(\frac{q^{2n}-1}{q-1}\right)\times\right.$$
$$\left.a_{2n}z^{2n}+\cdots\right]+\alpha\left[q\left(\frac{q^2-1}{q-1}\right)a_2z^2+q\left(\frac{q^3-1}{q-1}\right)\left(\frac{q^2-1}{q-1}\right)a_3z^3+q\left(\frac{q^4-1}{q-1}\right)\left(\frac{q^3-1}{q-1}\right)a_4z^4\right.$$
$$\left.+q\left(\frac{q^5-1}{q-1}\right)\left(\frac{q^4-1}{q-1}\right)a_5z^5+\cdots+q\left(\frac{q^{2n}-1}{q-1}\right)\left(\frac{q^{2n-1}-1}{q-1}\right)a_{2n}z^{2n}+\cdots\right]$$

$$=\left[(1-\alpha)\left(z+a_3z^3+a_5z^5+\cdots+a_{2n-1}z^{2n-1}+a_{2n+1}z^{2n+1}+\cdots\right)+\alpha\left(z+\left(\frac{q^3-1}{q-1}\right)a_3z^3\right.\right.$$
$$\left.\left.+\left(\frac{q^5-1}{q-1}\right)a_5z^5+\cdots+\left(\frac{q^{2n-1}-1}{q-1}\right)a_{2n-1}z^{2n-1}+\left(\frac{q^{2n+1}-1}{q-1}+\cdots\right)a_{2n+1}z^{2n+1}+\cdots\right)\right]$$
$$\left(1+P_1z+P_2z^2+P_3z^3+P_4z^4+...+P_{2n-1}z^{2n-1}+P_{2n}z^{2n}+...\right).$$

Equating the coefficients of like powers of z, we have

$$\left(\frac{q^2-1}{q-1}\right)(1+\alpha q)\,a_2=P_1. \tag{3.3}$$

$$q\left(\frac{q^2-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^3-1}{q-1}\right)\right)a_3=P_2. \tag{3.4}$$

$$\left(\frac{q^4-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^3-1}{q-1}\right)\right)a_4=P_3+P_1a_3\left(1-\alpha+\alpha\left(\frac{q^3-1}{q-1}\right)\right). \tag{3.5}$$

$$q\left(\frac{q^4-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^5-1}{q-1}\right)\right)a_5=P_4+P_2a_3\left(1-\alpha+\alpha\left(\frac{q^3-1}{q-1}\right)\right). \tag{3.6}$$

$$\begin{aligned}&\left(\frac{q^{2n}-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^{2n-1}-1}{q-1}\right)\right)a_{2n}=\\ &P_{2n-1}+\left[1-\alpha+\alpha\left(\frac{q^{2n-1}-1}{q-1}\right)\right]p_1a_{2n-1}+\cdots+\left[1-\alpha+\alpha\left(\frac{q^3-1}{q-1}\right)\right]a_3p_{2n-3}.\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\left(\frac{q^{2n}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2n+1}-1}{q-1}\right)\right)a_{2n+1}=\\ &P_{2n}+\left[1-\alpha+\alpha\left(\frac{q^{2n-1}-1}{q-1}\right)\right]p_2a_{2n-1}+\cdots+\left[1-\alpha+\alpha\left(\frac{q^3-1}{q-1}\right)\right]a_3p_{2n-2}.\end{aligned} \tag{3.8}$$

Using Lemma 2.2, we get

$$|a_2|\le\frac{A-B}{\left(\frac{q^2-1}{q-1}\right)(1+\alpha q)}. \tag{3.9}$$

$$|a_3|\le\frac{A-B}{q\left(\frac{q^2-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^3-1}{q-1}\right)\right)}. \tag{3.10}$$

$$|a_4|\le\frac{(A-B)\left(A-B+q\frac{q^2-1}{q-1}\right)}{q\left(\frac{q^2-1}{q-1}\right)\left(\frac{q^4-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^3-1}{q-1}\right)\right)}. \tag{3.11}$$

$$|a_5|\le\frac{(A-B)\left(A-B+q\frac{q^2-1}{q-1}\right)}{q^2\left(\frac{q^2-1}{q-1}\right)\left(\frac{q^4-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^5-1}{q-1}\right)\right)}. \tag{3.12}$$

It follows that (3.1) and (3.2) hold for $n=1,2$. We prove (3.1) using induction. Equation (3.7) in conjunction with Lemma (2.2) yield

$$|a_{2n}|\le\frac{(A-B)}{\left(\frac{q^{2n}-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^{2n-1}-1}{q-1}\right)\right)}\left[1+\sum_{k=1}^{n-1}\left(1-\alpha+\alpha\left(\frac{q^{2k+1}-1}{q-1}\right)\right)|a_{2k+1}|\right]. \tag{3.13}$$

We assume that (3.1) holds for $k=3,4,\ldots,(n-1)$. Then from(3.13), we obtain

(3.14)

$$|a_{2n}| \leq \frac{(A-B)}{\left(\frac{q^{2n}-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^{2n-1}-1}{q-1}\right)\right)}\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)^k\left(\frac{q^k-1}{q-1}\right)!}\prod_{j=1}^{k-1}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right)\right]$$

In order to complete the proof, it is sufficient to show that
(3.15)

$$\frac{(A-B)}{\left(\frac{q^{2m}-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^{2m-1}-1}{q-1}\right)\right)}\left[1+\sum_{k=1}^{m-1}\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)^k\left(\frac{q^k-1}{q-1}\right)!}\prod_{j=1}^{k-1}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right)\right]$$

$$=\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)^m\left(\frac{q^m-1}{q-1}\right)!\left(1+\alpha q\left(\frac{q^{2m-1}-1}{q-1}\right)\right)}\prod_{j=1}^{k-1}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right) \quad (m=3,\,4,\,5,\,\ldots)$$

(3.15) is valid for $m=3$.

Let us suppose that (3.15) is true for all $m,\ 3<m\leq(n-1)$. Then from (3.14)

$$\frac{(A-B)}{\left(\frac{q^{2n}-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^{2n-1}-1}{q-1}\right)\right)}\left[1+\sum_{k=1}^{m-1}\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)^k\left(\frac{q^k-1}{q-1}\right)!}\prod_{j=1}^{k-1}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right)\right]$$

$$=\left(\frac{q^{n-1}-1}{q^n-1}\right)\left[\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)\left(\frac{q^{n-1}-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^{2n-1}-1}{q-1}\right)\right)}\left(1+\sum_{k=1}^{n-2}\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)^k\left(\frac{q^k-1}{q-1}\right)!}\times\right.\right.$$

$$\left.\left.\prod_{j=1}^{k-1}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right)\right)\right]+\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)\left(\frac{q^n-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^{2n-1}-1}{q-1}\right)\right)}\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)^{n-1}\left(\frac{q^{n-1}-1}{q-1}\right)!}\times$$

$$\prod_{j=1}^{n-2}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right)=\left(\frac{q^{n-1}-1}{q^n-1}\right)\frac{(A-B)}{\left(1+\alpha q\left(\frac{q^{2n-1}-1}{q-1}\right)\right)\left(\frac{q^2-1}{q-1}\right)^{n-1}\left(\frac{q^{n-1}-1}{q-1}\right)!}\times$$

$$\prod_{j=1}^{n-2}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right)+\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)\left(\frac{q^n-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^{2n-1}-1}{q-1}\right)\right)}\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)^{n-1}\left(\frac{q^{n-1}-1}{q-1}\right)!}\times$$

$$\prod_{j=1}^{n-2}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right)=\frac{(A-B)}{\left(\frac{q^2-1}{q-1}\right)\left(\frac{q^n-1}{q-1}\right)\left(1+\alpha q\left(\frac{q^{2n-1}-1}{q-1}\right)\right)\left(\frac{q^2-1}{q-1}\right)^{n-1}\left(\frac{q^{n-1}-1}{q-1}\right)!}\times$$

$$\prod_{j=1}^{n-2}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right)\left(A-B+\left(\frac{q^2-1}{q-1}\right)\left(\frac{q^{n-1}-1}{q-1}\right)\right)=$$

$$\frac{(A-B)}{\left(1+\alpha q\left(\frac{q^{2n-1}-1}{q-1}\right)\right)\left(\frac{q^2-1}{q-1}\right)^n\left(\frac{q^n-1}{q-1}\right)!}\prod_{j=1}^{n-1}\left(A-B+q\left(\frac{q^{2j}-1}{q-1}\right)\right)$$

Thus (3.15) holds for $m=n$ and hence (3.1) follows. Similarly we can prove (3.2).

□

Theorem 3.2. *Let $f \in M_c^q(\alpha, A, B)$. Then for $n \geq 1, 0 \leq \alpha \leq 1$,*

$$|a_{2n}| \leq \frac{(A-B)}{q\left(\frac{q^{2n-1}-1}{q-1}\right)!\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)} \prod_{j=1}^{2n-2}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right) \tag{3.16}$$

$$|a_{2n+1}| \leq \frac{(A-B)}{q\left(\frac{q^{2n}-1}{q-1}\right)!\left(1-\alpha+\alpha\left(\frac{q^{2n+1}-1}{q-1}\right)\right)} \prod_{j=1}^{2n-2}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right) \tag{3.17}$$

Proof. From(1.2) and (1.4), we have

$$\Big[z+\left(\frac{q^2-1}{q-1}\right)a_2z^2+\left(\frac{q^3-1}{q-1}\right)a_3z^3+\left(\frac{q^4-1}{q-1}\right)a_4z^4+\left(\frac{q^5-1}{q-1}\right)a_5z^5+\cdots+\left(\frac{q^{2n}-1}{q-1}\right)\times$$
$$a_{2n}z^{2n}+\cdots\Big]+\alpha\Big[q\left(\frac{q^2-1}{q-1}\right)a_2z^2+q\left(\frac{q^3-1}{q-1}\right)\left(\frac{q^2-1}{q-1}\right)a_3z^3+q\left(\frac{q^4-1}{q-1}\right)\left(\frac{q^3-1}{q-1}\right)a_4z^4+$$
$$q\left(\frac{q^5-1}{q-1}\right)\left(\frac{q^4-1}{q-1}\right)a_5z^5+\cdots+q\left(\frac{q^{2n}-1}{q-1}\right)\left(\frac{q^{2n-1}-1}{q-1}\right)a_{2n}z^{2n}+\cdots\Big]=$$
$$\Big[(1-\alpha)\left(z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+\cdots+a_{2n}z^{2n}+\cdots\right)+\alpha\Big(z+\left(\frac{q^2-1}{q-1}\right)a_2z^2$$
$$+\left(\frac{q^3-1}{q-1}\right)a_3z^3+\left(\frac{q^4-1}{q-1}\right)a_4z^4+\left(\frac{q^5-1}{q-1}\right)a_5z^5+\cdots+\left(\frac{q^{2n}-1}{q-1}\right)a_{2n}z^{2n}+\cdots\Big)\Big]\times$$
$$\left(1+P_1z+P_2z^2+P_3z^3+P_4z^4+\cdots+P_{2n-1}z^{2n-1}+P_{2n}z^{2n}+\cdots\right)$$

Equating the coefficients of like powers of z, we have

$$q\left(1-\alpha+\alpha\left(\frac{q^2-1}{q-1}\right)\right)a_2=P_1 \tag{3.18}$$

$$q\left(\frac{q^2-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^3-1}{q-1}\right)\right)a_3=P_2+\left(1-\alpha+\alpha\left(\frac{q^2-1}{q-1}\right)\right)a_2P_1 \tag{3.19}$$

$$\begin{aligned} q\left(\frac{q^3-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^4-1}{q-1}\right)\right)a_4 &= P_3+\left(1-\alpha+\alpha\left(\frac{q^2-1}{q-1}\right)\right)a_2P_2 \\ &+\left(1-\alpha+\alpha\left(\frac{q^3-1}{q-1}\right)\right)P_1a_3 \end{aligned} \tag{3.20}$$

$$\begin{aligned} q\left(\frac{q^4-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^5-1}{q-1}\right)\right)a_5 &= P_4+\left(1-\alpha+\alpha\left(\frac{q^2-1}{q-1}\right)\right)a_2P_3 \\ +\left(1-\alpha+\alpha\left(\frac{q^3-1}{q-1}\right)\right)P_2a_3 &+\left(1-\alpha+\alpha\left(\frac{q^4-1}{q-1}\right)\right)P_1a_4 \end{aligned} \tag{3.21}$$

$$q\left(\frac{q^{2n-1}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)a_{2n}=P_{2n-1}+\left(1-\alpha+\alpha\left(\frac{q^{2}-1}{q-1}\right)\right)a_{2}P_{2n-2}+\cdots+\left(1-\alpha+\alpha\left(\frac{q^{2n-1}-1}{q-1}\right)\right)P_{1}a_{2n-1} \tag{3.22}$$

Using Lemma(2.2) and (3.18),(3.19),(3.20)and (3.21) we get

$$|a_2|\leq\frac{A-B}{q\left(1-\alpha+\alpha\left(\frac{q^2-1}{q-1}\right)\right)} \tag{3.23}$$

$$|a_3|\leq\frac{(A-B)\ (A-B+q)}{q^2\left(\frac{q^2-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^3-1}{q-1}\right)\right)} \tag{3.24}$$

$$|a_4|\leq\frac{(A-B)\ (A-B+q)\left(A-B+\left(\frac{q^2-1}{q-1}\right)\right)}{q^3\left(\frac{q^2-1}{q-1}\right)\left(\frac{q^3-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^4-1}{q-1}\right)\right)} \tag{3.25}$$

$$|a_5|\leq\frac{(A-B)\ (A-B+q)\left(A-B+q\left(\frac{q^2-1}{q-1}\right)\right)\left(A-B+q\left(\frac{q^3-1}{q-1}\right)\right)}{q^4\left(\frac{q^2-1}{q-1}\right)\left(\frac{q^3-1}{q-1}\right)\left(\frac{q^4-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^5-1}{q-1}\right)\right)}. \tag{3.26}$$

It follows that (3.16) hold for $n = 1, 2$. We now prove (3.16) using induction. Equation (3.22) in conjunction with Lemma 2.2 yields

$$|a_{2n}|\leq\frac{(A-B)}{q\left(\frac{q^{2n-1}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)}\times\left[1+\sum_{k=1}^{n-1}\left(1-\alpha+\alpha\left(\frac{q^{2k}-1}{q-1}\right)\right)|a_{2k}|+\sum_{k=1}^{n-1}\left(1-\alpha+\alpha\left(\frac{q^{2k+1}-1}{q-1}\right)\right)|a_{2k+1}|\right]. \tag{3.27}$$

We assume that (3.16) holds for $k = 3, 4, ..., (n-1)$. Then from(3.27), we obtain

$$|a_{2n}|\leq\frac{(A-B)}{q\left(\frac{q^{2n-1}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)}\times\left[1+\sum_{k=1}^{n-1}\frac{A-B}{\left(\frac{q^{2k-1}-1}{q-1}\right)!}\prod_{j=1}^{2k-2}\left(A-B+\left(\frac{q^{j}-1}{q-1}\right)\right)+\sum_{k=1}^{n-1}\frac{A-B}{\left(\frac{q^{2k}-1}{q-1}\right)!}\prod_{j=1}^{2k-1}\left(A-B+\left(\frac{q^{j}-1}{q-1}\right)\right)\right]. \tag{3.28}$$

In order to complete the proof, it is sufficient to show that
(3.29)

$$\frac{(A-B)}{q\left(\frac{q^{2m-1}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2m}-1}{q-1}\right)\right)}\left[1+\sum_{k=1}^{m-1}\frac{A-B}{\left(\frac{q^{2k-1}-1}{q-1}\right)!}\prod_{j=1}^{2k-2}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right)\right.$$
$$\left.+\sum_{k=1}^{m-1}\frac{A-B}{\left(\frac{q^{2k}-1}{q-1}\right)!}\prod_{j=1}^{2k-1}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right]\right]$$
$$=\frac{(A-B)}{q\left(\frac{q^{2m-1}-1}{q-1}\right)!\left(1-\alpha+\alpha\left(\frac{q^{2m}-1}{q-1}\right)\right)}\prod_{j=1}^{2m-2}\left[A-B+\left(\frac{q^j-1}{q-1}\right)\right],\qquad (m=3,4,5,\ldots).$$

(3.29) is valid for $m=3$. Let us suppose that (3.29) is true for all $m, 3<m\leq(n-1)$. Then from (3.28)

$$\frac{(A-B)}{q\left(\frac{q^{2n-1}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)}\left[1+\sum_{k=1}^{n-1}\frac{A-B}{\left(\frac{q^{2k-1}-1}{q-1}\right)!}\prod_{j=1}^{2k-2}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right)+\right.$$
$$\left.\sum_{k=1}^{n-1}\frac{A-B}{\left(\frac{q^{2k}-1}{q-1}\right)!}\prod_{j=1}^{2k-1}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right]\right]$$

$$=\left(\frac{q^{2n-3}-1}{q^{2n-1}-1}\right)\left(\frac{(A-B)}{q\left(\frac{q^{2n-3}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)}\times\left[1+\sum_{k=1}^{n-2}\frac{A-B}{\left(\frac{q^{2k-1}-1}{q-1}\right)!}\prod_{j=1}^{2k-2}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right)\right.\right.$$
$$\left.\left.+\sum_{k=1}^{n-2}\frac{A-B}{\left(\frac{q^{2k}-1}{q-1}\right)!}\prod_{j=1}^{2k-1}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right]\right]\right)$$

$$+\frac{(A-B)}{q\left(\frac{q^{2n-1}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)}\frac{A-B}{\left(\left(\frac{q^{2k-2}-1}{q-1}\right)!-1\right)}\prod_{j=1}^{2n-4}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right)$$
$$+\frac{(A-B)}{q\left(\frac{q^{2n-1}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)}\frac{A-B}{\left(\left(\frac{q^{2k-2}-1}{q-1}\right)!\right)}\prod_{j=1}^{2n-3}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right)$$
$$=\left(\frac{q^{2n-3}-1}{q^{2n-1}-1}\right)\frac{A-B}{\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)\left(\left(\frac{q^{2k-3}-1}{q-1}\right)!\right)}\prod_{j=1}^{2n-4}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right)$$
$$+\frac{(A-B)}{q\left(\frac{q^{2n-1}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)}\frac{A-B}{\left(\left(\frac{q^{2k-3}-1}{q-1}\right)!\right)}\prod_{j=1}^{2n-4}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right)$$
$$+\frac{(A-B)}{q\left(\frac{q^{2n-1}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)}\frac{A-B}{\left(\left(\frac{q^{2k-2}-1}{q-1}\right)!\right)}\prod_{j=1}^{2n-3}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right)$$

$$= \frac{(A-B)}{q\left(\frac{q^{2n-1}-1}{q-1}\right)\left(\left(\frac{q^{2k-3}-1}{q-1}\right)!\right)\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)} \prod_{j=1}^{2n-4}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right) \times$$
$$\left(A-B+\left(\frac{q^{2n-3}-1}{q-1}\right)\right) + \frac{(A-B)}{q\left(\frac{q^{2n-1}-1}{q-1}\right)\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)} \frac{A-B}{\left(\left(\frac{q^{2k-2}-1}{q-1}\right)!\right)} \times$$
$$\prod_{j=1}^{2n-3}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right) = \frac{(A-B)}{q\left(\frac{q^{2n-1}-1}{q-1}\right)!\left(1-\alpha+\alpha\left(\frac{q^{2n}-1}{q-1}\right)\right)} \prod_{j=1}^{2n-2}\left(A-B+\left(\frac{q^j-1}{q-1}\right)\right)$$

Thus (3.29) holds for $m = n$ and hence (3.16) follows. Similarly we can prove (3.17). □

On specializing the values of α in Theorem 3.1 and Theorem 3.1, we get the following.

Remark 3.1. In Theorem 3.1, if we let $\alpha = 0$ and $q \to 1^-$, we get the coefficient inequality of the starlike functions with respect to symmetric points and if we set $\alpha = 1$ and $q \to 1^-$, we get the coefficient inequality of the convex functions with respect to symmetric points.

Remark 3.2. In Theorem 3.2, if we let $\alpha = 0$ and $q \to 1^-$, we get the similar result obtained for starlike functions with respect to conjugate points and if we set $\alpha = 1$ and $q \to 1^-$, we get the coefficient inequality of convex functions with respect to conjugate points. For other values of α the transition is smooth.

Remark 3.3. If we let $q \to 1^-$ in Theorem 3.1 and Theorem 3.2, we get the results obtained by Selvaraj and Vasanthi [13].

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