A discussion on analytical study of Semi-closed set in topological space

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Abstract : In this paper, we introduce a new class of sets in the topological space, namely Semi-closed sets in the topological space. We find characterizations of these sets. Further, we study some fundamental properties of Semi-closed sets in the topological space.

Keywords : Open set, Closed set, Interior of a set & Closure of a set.

I. Introduction

The term Semi-closed set which is a weak form of closed set in a topological space and it is introduced and defined by the mathematician N. Biswas [10] in the year 1969. The term Semi-closure of a set in a topological space defined and introduced by two mathematician Crossley S. G. & Hildebrand S. K. [3,4] in the year 1971. The mathematician N. Levine [1] also defined and studied the term generalized closed sets in the topological space in Jan 1970. The term Semi-Interior point & Semi-Limit point of a subset of a topological space was defined and studied by the mathematician P. Das [9] in the year 1973 and he was also defined and studied the term Semi-derived set of a subset of a topological space.

In this paper, analogous to N. Biswas's [10] Semi-closed set, we investigate the certain results of Semi-closed set in the topological space.

II. Preliminaries

Throughout this paper (X , τ) is always denote topological spaces on which no separation axioms are assumed, unless otherwise mentioned. When A is a subset of (X , τ) then $C_L(A)$ & $I_N(A)$ are denote the closure and interior of the set A in the topological space.

2.1 . <u>Semi-closed set</u> [10] : Let (X , τ) be a topological space of a nonempty set X with the topology τ and let A ⊆ X then the set A is said to be **Semi-closed set** iff ∃ a closed set F s. t. $I_N(F)$ ⊆ A ⊆ F. It is denoted by **S. C.(X)**.

Also, the set A is said to be Semi-closed set iff A^c is Semi-open set in the topological space.

The family of all Semi-closed set in the topological space is denoted by **S.C.(X)**.

2.2 . <u>Semi-open set</u> [1] : Let (X **,** τ) be a topological space of a nonempty set X with topology τ & let A ⊆ X then the set A is **Semi-open set** iff ∃ an open set G s. t. G ⊆ A ⊆ C_L [G].

2.3 . <u>Proposition</u> : The set of real space R with usual topology τ is always Semi-closed set.

We shall show it by taking an example with the usual topology.

Let X = R ; (the real space) and since the set of real space R is a closed set in the topological space so ∃ a closed set R s. t. I_N [R] ⊆ R ⊆ R and I_N [R] = R. So, R ⊆ R ⊆ R.

Hence, the set R with the usual topology τ is always a Semi-closed set in the topological space.

2.4 . <u>Proposition</u> : The empty set φ is always a Semi-closed set in the real space with the usual topology.

We shall show it by taking an example in the real space R with the usual topology.

Let X = φ ; (the empty set) and since the set φ is always a closed set in the topological space. So, ∃ a closed set φ s. t. I_N(φ) ⊆ φ ⊆ φ & I_N(φ) = φ. So, φ ⊆ φ ⊆ φ.

Hence, the set φ is always a Semi-closed set in the real space with the usual topology.

2.5 . <u>Proposition</u> : Every open set is always a Semi-closed set in the real space with the usual topology.

We shall show it by taking an example in the real space R with the usual topology.

Let X =]a , b[; (open interval) & since every closed interval is a closed set in the topological space. So, ∃ a closed set [a , b] s. t. I_N([a , b]) ⊆]a , b[⊆ [a , b] and I_N [{[a , b]}] =] a , b [. So,]a , b[⊆]a , b[⊆ [a , b].

Hence, every open interval is always a Semi-closed set in the real space with the usual topology. As we know that every open interval is an open set in the real space. **Therefore, every open set is always a Semi-closed set in the real space with the usual topology.**

2.6. <u>Proposition</u> : Every closed set is always a Semi-closed set in the real space with the usual topology.

We shall show it by taking an example in real space R with the usual topology.

Let X = [a , b] ; (closed interval) & since every closed interval is a closed set in the topological space so ∃ a closed set [a , b] s. t. I_N([a , b]) ⊆ [a , b] ⊆ [a , b] & I_N([a , b]) =] a , b [.

So,]a , b[⊆ [a , b] ⊆ [a , b].

Hence, every closed interval is always a Semi-closed set in the real space with the usual topology. As we know that every closed interval is always a closed set in the real space. **Therefore, every closed set is always a Semi-closed set in the real space with the usual topology.**

2.7 . <u>Proposition</u> : Every half open interval is always a Semi-closed set in the real space with the usual topology.

We shall show it by taking an example in real space R with the usual topology.

<u>Case [1]</u> : Consider X = [a , b [; (half open interval)

Let X = [a , b [& since every closed interval is a closed set in the topological space so ∃ a closed set [a , b] s. t. I_N([a , b]) ⊆ [a , b [⊆ [a , b] & I_N([a , b] =]a , b[.

So,]a , b[⊆ [a , b [⊆ [a , b].

Hence, X = [a , b [is always a Semi-closed set in the real space with the usual topology.

<u>Case [2]</u> : Consider X =] a , b] ; (half open interval)

Let X =]a , b] & since every closed interval is a closed set in the topological space. So, ∃ a closed set [a , b] s. t. I_N ([a , b]) ⊆] a , b] ⊆ [a , b] & I_N([a , b]) =]a , b[.

So,]a , b[⊆] a , b] ⊆ [a , b].

Hence, X =] a , b] is always a Semi-closed set in the real space with the usual topology.

Therefore, every half open interval is always a Semi-closed set in the real space with the usual topology.

2.8 . <u>Theorem</u> : Let (X , τ) be a topological space of a nonempty set X with the topology τ and let A ⊆ X then the set A is Semi-closed set iff $I_N[\{C_L(A)\}] \subseteq$ A.

<u>Proof</u> :

Necessary : Let the set A be Semi-closed set in the set X then ∃ a closed set F s. t. $I_N[F] \subseteq A \subseteq F$, for some closed set F. But, $C_L(A) \subseteq F$. Thus, $I_N[\{C_L(A)\}] \subseteq I_N[F]$. Hence, $I_N[\{C_L(A)\}] \subseteq I_N[F] \subseteq A \subseteq F \implies$ **$I_N[\{C_L(A)\}] \subseteq A$.**

Sufficient : Let $I_N[\{C_L(A)\}] \subseteq A$ & $I_N[\{C_L(A)\}] \subseteq A \subseteq C_L(A)$ and also let $F = C_L(A)$, i.e., $I_N[F] \subseteq A \subseteq F$, for some closed set F. **Thus, the set A is Semi-closed set in X.**

2.9 . Theorem : Let (X , τ) be a topological space of a nonempty set X with topology τ & let $A \subseteq X$ and let the set A is Semi-closed set and suppose $I_N(A) \subseteq G \subseteq A$ then the set G is also a Semi-closed set in the topological space.

Proof : Since, given the set A is a Semi-closed set. So, ∃ a closed set F s. t. $I_N(F) \subseteq A \subseteq F$ and given $I_N(A) \subseteq G \subseteq A$ then $I_N(A) \subseteq G \subseteq A \subseteq F$. But, $I_N(A) \subseteq I_N(F)$.

So, $I_N(A) \subseteq I_N(F) \subseteq G \implies I_N(F) \subseteq G$. **Thus, $I_N(F) \subseteq G \subseteq F$.**

Hence, the set G is also Semi-closed set in the topological space.

2.10 . Remark : The union of two Semi-closed set is may or may not be a Semi-closed set in the topological space.

We shall show it by taking an example in the real space R with the usual topology.

Case - [1] : In the following 1st example, the union of two Semi-closed set is not a Semi-closed set in the topological space.

We have discussed earlier that every open interval is the Semi-closed set.

Let $F =]a , b[\cup]b , c[$; where $a < b < c$.

So, $F =]a , b[\cup]b , c[= \{]a , c[- \{b\}\}$ then $C_L \{F\} = C_L [\{]a , c[- \{b\}\}] = [a , c]$ and

$I_N [\{C_L (F)\}] = I_N [\{ [a , c]\}] =] a , c [$ & $] a , c[\not\subseteq \{]a , c[- \{b\}\} = F$

Hence, $I_N [C_L \{ (F) \}] \not\subseteq F$.

Thus, the union of two Semi-closed set, i.e., $F =] a , b[\cup] b , c[$; $a < b < c$ is not a Semi-closed set in the topological space.

Case - [2] : In the following 2nd example, the union of two Semi-closed set is a Semi-closed set in the topological space.

We have discussed earlier that the every closed interval is the Semi-closed set.

Let $F = [a , b] \cup [b , c]$; where, $a < b < c$.

So, $F = [a , b] \cup [b , c] = [a , c]$ then $C_L (F) = C_L [\{ [a , c] \}] = [a , c]$

& $I_N [\{C_L(F)\}] = I_N [\{]a , c[\}] =]a , c [$ & $]a , c[\subseteq [a , b] \cup [b , c]$; $a < b < c$.

Because, $]a , c[\subseteq [a , c]$. **Hence, $I_N [C_L\{ (F)\}] \subseteq F$.**

Thus, the union of two Semi-closed set, i.e., F = [a , b] U [b , c] is a Semi-closed set in the topological space.

Hence, in any condition the union of two Semi-closed set is may or may not be a Semi-closed set in the topological space.

2.11 . Remark : The intersection of two Semi-closed set is always a Semi-closed set in the topological space.

We shall show it by taking an example in real space R with the usual topology.

Case -[1] : In the following 1st example, the intersection of two Semi-closed set is a Semi-closed set in the topological space.

We have discussed earlier that every open interval is the Semi-closed set.

Let F =]a , b[∩]b , c[; where, a < b < c.

So, F =]a , b[∩]b , c[= φ then C_L (F) = C_L[{φ}] = φ & I_N[{C_L(F)}] = I_N[{φ}] = φ & φ ⊆ φ. **Hence, I_N [C_L{(F)}] ⊆ F.**

Thus, the intersection of two Semi-closed set, i.e., F =]a , b[∩]b , c[; a < b < c is a Semi-closed set in the topological space.

Case - [2] : In the following 2nd example, the intersection of two Semi-closed set is also a Semi-closed set in the topological space.

We have discussed earlier that every closed interval is the Semi-closed set.

Let F = [a , b] ∩ [b , c] ; where, a < b < c.

So, F = [a , b] ∩ [b , c] = {b} then C_L (F) = C_L [{b}] = φ & I_N [{C_L(F)}] = I_N [{φ}] = φ and φ ⊆ {b}; i.e., φ ⊆ F = {b}. **Hence, I_N [C_L{(F)}] ⊆ F.**

Thus, the intersection of two Semi-closed set, i.e., F = [a , b] ∩ [b , c] is always a Semi-closed set in the topological space.

Hence, in any condition the intersection of two Semi-closed set is always a Semi-closed set in the topological space.

2.12 . Proposition : The intersection of arbitrary collection of Semi-closed set is always a Semi-closed set in the topological space.

Or, let $\{ A_\alpha \}_{\alpha \in \Delta}$ be the collection of all Semi-closed set then $\bigcap_{\alpha \in \Delta} \{ A_\alpha \}$ is always a Semi-closed set in the topological space.

Verification :

Since, every A_α is a Semi-closed set in the topological space so for every $\alpha \in \Delta$, $\exists$ a closed set G_α s. t. $I_N(G_\alpha) \subseteq A_\alpha \subseteq G_\alpha$ then $\bigcap_{\alpha \in \Delta} \{I_N(G_\alpha)\} \subseteq \bigcap_{\alpha \in \Delta} \{A_\alpha\} \subseteq \bigcap_{\alpha \in \Delta} \{G_\alpha\}$.

But, $I_N[\{\bigcap_{\alpha \in \Delta} \{G_\alpha\}\}] \subseteq \bigcap_{\alpha \in \Delta} (G_\alpha)$. If $G = \bigcap_{\alpha \in \Delta} \{G_\alpha\}$ & $A = \bigcap_{\alpha \in \Delta} \{A_\alpha\}$ then $\exists$ a closed set G s. t. the set A satisfies $\mathbf{I_N(G) \subseteq A \subseteq G}$.

Hence, the set A = $\bigcap_{\alpha \in \Delta} \{A_\alpha\}$ is Semi-closed set in the topological space.

Therefore, the intersection of an arbitrary family of Semi-closed sets is Semi-closed set in the topological space.

2.13 . <u>Semi-Limit point</u> [9] : Let (X, τ) be a topological space of a nonempty set X with the topology τ & let $A \subseteq X$ then a point $x \in X$ is called a **Semi-Limit point (or a Semi-Cluster point or a Semi-Accumulation point) of a set A** iff every Semi-open set G of the point 'x' contains a point of A different from the point x. It is denoted by **SL(A)**.

i.e., either $(G \cap A) - \{x\} \neq \varphi$ or $[G - \{x\}] \cap A \neq \varphi$; $\forall$ Semi-open set G of x.

Also, $(G \cap A) \neq \varphi$ and $(G \cap A) \neq \{x\}$.

2.14 . <u>Semi-Adherent point</u> [9] : Let (X, τ) be a topological space of a nonempty set X with the topology τ & let $A \subseteq X$ then a point $x \in X$ is called a **Semi-Adherent point (or Semi-Contact point) of a set A** iff every Semi-open set G of the point 'x' contains a point of A, i.e., $(G \cap A) \neq \varphi$; $\forall$ Semi-open set G of x.

Also, The set of all Semi-Adherent points of the set A is called the Semi-Adherence of the set A. It is denoted by **SAdh[A]**.

2.15 . <u>Semi-Derived set</u> [9] : The set of all Semi-Limit points of the set A is called the **Semi-Derived set of the set A**. It is denoted by **SDe[A]**.

2.16 . <u>Semi-closure of a set</u> : Let (X, τ) be a topological space of a nonempty set X with the topology τ & Let $A \subseteq X$ in the topological space (X, τ) then **Semi-closure of a set A is the intersection of all Semi-closed set containing A.**

Also, the Semi-closure of a set A is the set of all Semi-Adherent points of a set in the topological space. It is denoted by **$SC_L[A]$**.

2.17 . <u>Semi-Adherent point</u> : Let (X, τ) be a topological space of a nonempty set X with topology τ & let $A \subseteq X$ then a point $x \in X$ is called **a Semi-Adherent point of a set A iff x is a point of A Or x is a Semi-Limit point of A.**

Accordingly, $SC_L[A] = A \cup SDe[A]$.

The Semi-closed set can be characterized in terms of the Semi-Limit points in the same way as Closed sets in terms of Limit points.

2.18 . <u>Semi-closed set</u> : Let (X, τ) be a topological space of a nonempty set X with topology τ & let A ⊆ X then the set A is said to be **Semi-closed set if the set A contains all its Semi-Limit points, i.e., SDe[A] ⊆ A.**

We now written several essential properties of semi-closure of sets as follows :

2.19. <u>Properties of Semi-closure of a set</u> :

Let (X , τ) be a topological space of a nonempty set X with the topology τ & let A , B ⊆ X in topological space (X , J) then

[1] . $SC_L[\varphi] = \varphi$ & $SC_L[X] = X$

[2] . $A \subseteq SC_L[A]$

[3] . If $A \subseteq B$, then $SC_L[A] \subseteq SC_L[B]$

[4] . $SC_L[A \cup B] \subseteq SC_L[A] \cup SC_L[B]$; ∀ A , B ⊆ X and equality may not hold

[5] . $SC_L[A \cap B] \subseteq SC_L[A] \cap SC_L[B]$; ∀ A , B ⊆ X and equality may not hold

[6] . $SC_L\{SC_L[A]\} = A$; ∀ A ⊆ X

[7] . The set A is Semi-closed set iff $A = SC_L[A]$

[8] .The Semi-closure of the set A is the smallest Semi-closed set containing set A.

We prove the equality may not hold in (4), follows from the following counter e. g.

Let X = { a , b , c , d , e } & topology τ = { φ , X , {a} , {c , d} , {a , c , d} , {b , c , d , e} }.

Consider set A = {a} and B = {b} then $SC_L(A \cup B)$ = { a , b , c , d }, $SC_L(A)$ = {a} , $SC_L(B)$ = { b , c } & $SC_L(A) \cup SC_L(B)$ = { a , b , c }. But, { a , b, c, d } ≠ { a , b , c }.

Hence, $SC_L[A \cup B] \neq SC_L[A] \cup SC_L[B]$.

Again, we prove the equality may not hold in (5), follows from the following counter e. g.

Let X = { a , b , c , d , e } & topology τ = { φ , X , {a}, {c , d}, {a , c , d}, {b , c , d , e} }.

Consider set A = {a , b} and B = {b , d} then (A ∩ B) = { b } & $SC_L(A \cap B)$ = { b , c } , $SC_L(A)$ = { a , b , c , d} & $SC_L(B)$ = { b , c , d } then $SC_L(A) \cap SC_L(B)$ = { b , c , d }.

But, { b , c } ≠ { b , c , d }.

Hence, $SC_L[A \cap B] \neq SC_L[A] \cap SC_L[B]$.

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