GEODETIC CERTIFIED DOMINATION NUMBER OF A GRAPH

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ABSTRACT

A set S of vertices in a connected graph G = (V, E) is called a geodetic set if every vertex not in S lies on a shortest path between two vertices from S. A set S of vertices in G is called a dominating set of G if every vertex not in S has at least one neighbor in S. A subset S ⊆ V(G) is called a certified dominating set of G if S is a dominating set of G and every vertex in S has either zero or at least two neighbors in V − S. A set S is called a geodetic certified dominating set of G if S is both geodetic and certified dominating set of G. The geodetic certified domination number is the minimum cardinality of a geodetic certified dominating set of G. In this paper we introduced and studied the geodetic certified domination number of certain classes of graphs. Also, some of us general properties are studied. It is shown that for any two integers 3 ≤ a ≤ p, a ≠ p − 1, there exists a connected graph G with γg(G) = a and |V(G)| = p.

Keywords: Dominating set, Geodetic number, Certified domination, Geodetic certified domination.

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1. INTRODUCTION

By a graph \( G = (V, E) \) we mean a finite, connected, undirected graph with neither loops nor multiple edges. The order \( |V| \) and size \( |E| \) of \( G \) and denoted by \( p \) and \( q \), respectively. For graph theoretic terminology we refer to west [7]. A vertex \( v \) of \( G \) is said to be an extreme vertex if the subgraph induced by its neighborhood is complete. The set of all extreme vertices is denoted by \( \text{Ext}(G) \). Two vertices \( u \) and \( v \) are said to be an antipodal vertices if \( d(u, v) = \text{diam}(G) \). If \( e = \{u,v\} \) is an edge of a graph \( G \) with \( \deg(u) = 1 \) and \( \deg(v) > 1 \), then we call \( e \) as a pendant edge, \( u \) a pendant vertex or end vertex and \( v \) a support vertex. An \( x – y \) path of length \( d(x, y) \) is called geodesic. A vertex \( v \) is said to lie on an geodesic \( P \) if \( v \) is an internal vertex of \( P \). The closed interval consists of \( x, y \) and all vertices lying on some \( x – y \) geodesic of \( G \) and for a non-empty set \( S \subseteq V(G) \), \( I[S] = \bigcup_{x,y \in S} I[x,y] \). A set \( S \subseteq V(G) \) in a connected graph is called a geodetic set of \( G \) if \( I[S] = V(G) \). The geodetic number of \( G \), denoted by \( g(G) \), is the minimum cardinality of a geodetic sets of \( G \). Various concepts inspired by geodetic sets are introduced in [1, 2].

A set \( S \subseteq V(G) \) is called a dominating set of \( G \) if for every vertex \( v \in V – S \), there exists a vertex \( u \in S \) such that \( v \) is adjacent to \( u \). A subset \( S \subseteq V(G) \) is called a certified dominating set of \( G \) if \( S \) is a dominating set of \( G \) and every vertex in \( S \) has either zero or atleast two neighbors in \( V – S \). The certified domination number of \( G \), denoted by \( \gamma_{\text{cer}}(G) \), is the minimum cardinality of a certified dominating set of \( G \) [3].

It is easily seen that a certified dominating set is not in general is a geodetic set of \( G \). Also the converse is not a valid in general. This has motivated as to study the new domination. Concept of Geodetic certified domination. This parameters has some possible relations in social networks.

2. BASIC RESULTS

Theorem 2.1. [2] Each extreme vertex of a connected graph belong to every geodetic set of \( G \).

Theorem 2.2. [5] A vertex \( v \) of a connected graph \( G \) is a cut vertex of \( G \) if and only if there exists vertices \( u \) and \( w \) distinct from \( v \) such that \( v \) lies on every \( u – w \) path of \( G \).
3. GEODETIC CERTIFIED DOMINATION NUMBER OF A GRAPH

**Definition 3.1.** Let $G = (V, E)$ be a connected graph with at least two vertices. A subset $S \subseteq V(G)$ is called a geodetic certified dominating set of $G$ if $S$ is both geodetic and certified dominating set of $G$. A geodetic certified dominating set of minimum cardinality is a minimum geodetic certified dominating set and this cardinality is the geodetic certified domination number $c_{\gamma_g}(G)$ of $G$.

**Example 3.2.** For the graph $G$ in Figure 3.1. It is easily checked that a 2-element subset is a geodetic certified dominating set. Since $S = \{v_1, v_3, v_5\}$ is the unique minimum geodetic certified dominating set of $G$. Thus, $c_{\gamma_g}(G) = 3$

**Figure 3.1**

**Remark 3.3.** For the graph $G$ gives in figure 3. 1. $S = \{v_1, v_5\}$ is a unique minimum geodetic set of $G$ and so $g(G) = 2$. Also $S_1 = \{v_1, v_3\}$ is the unique minimum certified dominating set of $G$. Therefore, $\gamma_{cer}(G) = 2$. Thus, the geodetic number and the certified domination number.

**Remark 3.4.** For any graph $G$, there can be more than one geodetic certified dominating set. For the graph $G$ in Figure 3.2, $S = \{v_1, v_4\}$ is a minimum geodetic certified dominating set of $G$. Also $S_1 = \{v_3, v_6\}$ is also a minimum geodetic certified dominating set of $G$. Thus, there can be more than one minimum geodetic certified dominating set for a connected graph $G$. 


Remark 3.5. There is no relation connecting $g(G)$ and $\gamma_{\text{cer}}(G)$. Consider the graph $G$ given in Figure 3.3.

The graph $G$ with $g(G) > \gamma_{\text{cer}}(G)$. Here, $S = \{v_1, v_3, v_4\}$ is a minimum geodetic set of $G$ and so $g(G) = 3$. But it is easily verified that $S_1 = \{v_1, v_3\}$ is a minimum certified dominating set of $G$ and so $\gamma_{\text{cer}}(G) = 2$. Therefore, $g(G) > \gamma_{\text{cer}}(G)$. Now, consider the graph $G$ in Figure 3.4.
The graph $G$ with $\gamma_{cer}(G) > g(G)$. Here $S = \{v_2, v_4, v_6\}$ is a minimum certified dominating set of $G$ and so $\gamma_{cer}(G) = 3$. But it is clear that $S_1 = \{v_1, v_7\}$ is a minimum geodetic set of $G$ and so $g(G) = 2$. Therefore, $\gamma_{cer}(G) > g(G)$.

**Theorem 3.6.** For any connected graph $G$ of order $p$, $2 \leq \max \{g(G), \gamma_{cer}(G)\} \leq c\gamma_g(G) \leq p$.

**Proof.** Any geodetic set contains minimum two vertices and so $\max \{g(G), \gamma_{cer}(G)\} \geq 2$. Also by Remark 3.5, there is no relation connecting $g(G)$ and $\gamma_{cer}(G)$. But every geodetic certified dominating set is also a geodetic set and a certified dominating set of $G$. Thus, $\max \{g(G), \gamma_{cer}(G)\} \leq c\gamma_g(G)$. Moreover, since $G$ is connected, it is obvious that every vertices in $G$ forms a geodetic certified dominating set of $G$ and so $c\gamma_g(G) \leq p$. Hence $2 \leq \max \{g(G), \gamma_{cer}(G)\} \leq c\gamma_g(G) \leq p$.

**Observation 3.7**

(i) Every geodetic certified dominating set of $G$ contains all its pendant vertices.

(ii) Every superset of a geodetic certified dominating set of $G$ need not be a geodetic certified dominating set of $G$.

For example, Consider the graph $G$ in Figure 3.3. It is easily verified that $S = \{v_1, v_3, v_4, v_5\}$ is a geodetic certified dominating set of $G$. But for every $v \in V - S$, $S \cup \{v\}$ is not a geodetic certified dominating set of $G$.

**Theorem 3.8** Each extreme vertex of a connected graph $G$ belong to every geodetic certified dominating set of $G$.

**Proof.** Since every geodetic certified dominating set of $G$ is also a geodetic set, the result follows from Theorem 2.1.

**Corollary 3.9.** For any connected graph $G$ with $m$ – extreme vertices, then

$c\gamma_g(G) \geq \max\{2, m\}$.

**Proof.** This follows from Theorem 3.6 and Theorem 3.8.
Theorem 3.10. Let G be a connected graph with cut vertices and let S be a geodetic certified dominating set of G. If v is a cut vertex of G, then every component of G – v contains an element of S.

Proof. Let v be a cut – vertex of G and S be a geodetic certified dominating set of G. Suppose that there exists a component, say H of G – v such that H contains no vertex of S. By Theorem 3.8, S contains all its extreme vertices of G. This shows that H does not contain any extreme vertex of G. Let u ∈ V(H). (This is possible since G is connected). Since S is ageodetic certified dominating set of G, there exists vertices x, y ∈ S such that u is an internal vertex of some x – y geodesic P : x = u₀, u₁, ..., uₙ, ..., u₁ = y in G. Since v is a cut vertex of G, by Theorem 2. 2, the x – u subpath of P and u – y subpath of P both contains v. It follows that P is not a path, which is a contradiction. Hence, every component of G – v contains an element of S.

Corollary 3.11. Let G be a connected graph with cut vertices and let S be a geodetic certified dominating set of G. Then every branch of G contains an element of S.

Theorem 3.12. Each support vertex of a connected graph G belongs to every geodetic certified dominating set of G.

Proof. Let S be a geodetic certified dominating set of G and let v be a support vertex of G. Let u₁, u₂, ..., uₙ(≥ 1) be the end vertices adjacent with v in G. Suppose v ∉ S. By Theorem 3.8. u₁, u₂, ..., uₙ ∈ S. Then u₁, u₂, ..., uₙ has exactly one neighbor v in V – S. It follows that S is not a geodetic certified dominating set of G. Here, v ∈ S.

Corollary 3.13. Let G be a connected graph. If G contains only the extreme and support vertices. Then cγ₉(G) = p.

Proof. This follows from Theorem 3.8 and Theorem 3.12.

Theorem 3.14. For p ≥ 4 and p ≠ S, cγ₉(Cₚ) = γ₉(Cₚ) = \left\lceil \frac{p}{2} \right\rceil.

Proof. Let Cₚ = (v₁, v₂, ..., vₚ, v₁).

Case 1. p = 4

In this case, every certified dominating set is geodetic. Hence, cγ₉(C₄) = γ₉(C₄) = 2.
Case 2. P > 4 and p \neq S

In this case, if \( p \equiv 0 \pmod{3} \) or \( p \equiv 1 \pmod{3} \) or \( p \equiv 2 \pmod{3} \), then \( S = \{v_1, v_4, ..., v_{p-2}\} \) or \( S = \{v_1, v_2, v_6, v_9, ..., v_{p-4}, v_{p-2}\} \) or \( S = \{v_1, v_4, ..., v_{p-4}, v_{p-1}\} \) is a minimum certified dominating set of \( C_p \), respectively. Also, \( S \) is a geodetic set. Thus, \( c\gamma_g(C_p) \leq \gamma_{cer}(C_p) \). By Theorem 3.6, we conclude \( c\gamma_g(C_p) = \gamma_{cer}(C_p) \).

**Theorem 3.15.** For \( p \geq 5 \), \( c\gamma_g(W_p) = g(W_p) = \left\lceil \frac{p-1}{2} \right\rceil \).

**Proof.** Let \( p \geq 5 \) and \( V(W_p) = \{v_0, v_1, v_2, ..., v_{p-1}\} \), where \( \deg(v) = p - 1 \) and for \( 1 \leq i \leq p - 1 \), \( \deg(v_i) = 3 \).

Case 1. \( p \) is even.

Then \( S = \{v_1, v_3, ..., v_{p-3}, v_{p-1}\} \) is a minimum geodetic set of \( W_p \). Clearly \( S \) is independent and each vertex in \( S \) is adjacent to \( v \). This \( S \) is a certified dominating set of \( W_p \). Therefore, \( S \) is a geodetic certified dominating set of \( W_p \) and so \( c\gamma_g(W_p) \leq g(W_p) \). By Theorem 3.6, we conclude \( c\gamma_g(W_p) = g(W_p) \).

Case 2. \( p \) is odd.

Then \( S_1 = \{v_1, v_3, ..., v_{p-4}, v_{p-2}\} \) is a minimum geodetic set of \( W_p \). As similar as in case 1, that \( S_1 \) is a minimum geodetic certified dominating set of \( W_p \) and so \( c\gamma_g(W_p) = g(W_p) = \left\lceil \frac{p-1}{2} \right\rceil \).

**Observation 3.16.** The geodetic certified domination number of some standard graphs can be easily found, and are given as follows.

(i) For the path graph \( P_p \), \( c\gamma_g(P_p) = p \).

(ii) For the complete graph \( K_p \), \( c\gamma_g(K_p) = p \).

(iii) For the star graph \( K_{1,p-1} \), \( c\gamma_g(K_{1,p-1}) = p \).

(iv) For the complete bipartite graph \( K_{p,q} \) with \( p, q \geq 2 \), \( c\gamma_g(K_{p,q}) = \min\{p, q, 4\} \).

(v) For the complement of a path graph \( \overline{P_p} \) with \( p \geq 5 \), \( c\gamma_g(\overline{P_p}) = 3 \).
(vi) For the complement of a cycle graph $\overline{C_p}$ with $p \geq 5$, $c_{\gamma g}(\overline{C_p}) = 3$.

(vii) For the $k$-cube graph $Q_k$ with $k \geq 3$, $c_{\gamma g}(Q_k) = 2^{k-2}$.

The following Theorem characterizes graph for which the geodetic certified domination number is 2.

**Theorem 3.17.** Let $G$ be a connected graph of order $p \geq 2$. Then $c_{\gamma g}(G) = 2$ if and only if $G = K_2$ or $G$ has a geodetic set $S = \{x, y\}$ with $\deg(x), \deg(y) \geq 2$ and $d(x, y) \leq 3$.

**Proof.** Let $G$ be a connected graph of order $p \geq 2$. If $G = K_2$, then by observation 3.16(ii), $c_{\gamma g}(G) = 2$. Now, assume $G \neq K_2$. Then $G$ has a geodetic set $S = \{x, y\}$ with $\deg(x), \deg(y) \geq 2$ and $d(x, y) \leq 3$. We show that $S$ is a certified dominating set of $G$.

Since $d(x, y) \leq 3$, it is clear that $S$ is a dominating set of $G$. Now, we claim that every vertex in $S$ has either zero or at least two neighbors in $V - S$. Since $S$ is a geodetic set of $G$ and $G \neq K_2$, $S$ has at least one neighbor in $V - S$. Suppose $S$ has exactly one neighbor in $V - S$. Then $u$ is adjacent to exactly one vertex in $V - S$. It follows that $\deg(u) = 1$. Similarly, $\deg(v) = 1$, which is a contradiction. Hence $S$ is a certified dominating set of $G$ and so $c_{\gamma g}(G) \leq |S| = 2$. By Theorem 3.6, we conclude, $c_{\gamma g}(G) = 2$.

Conversely, assume $c_{\gamma g}(G) = 2$ and let $S = \{x, y\}$ be a minimum geodetic certified dominating set of $G$. Then two cases arise.

Case 1. $x$ and $y$ are adjacent in $G$. Since $S$ is a geodetic set, then $G \cong K_2$.

Case 2. $x$ and $y$ are non-adjacent in $G$. Since $S$ is a certified dominating set of $G$, every vertex in $S$ has at least two neighbors in $V - S$. It follows that $d(x, y) \leq 3$ and $\deg(x), \deg(y) \geq 2$. Thus, there exists a geodetic set $S = \{x, y\}$ of $G$, such that $d(x, y) \leq 3$ and $\deg(x), \deg(y) \geq 2$.

**Theorem 3.18.** Let $G$ be a connected graph with $|\text{Ext}(G)| \geq 2$. If every pair of extreme vertices are connected by a unique geodesic path, then $c_{\gamma g}(G) = p$.

**Proof.** Let $S$ be a minimum geodetic certified dominating set of $G$. Since $|\text{Ext}(G)| \geq 2$, $G$ has at least two extreme vertices. By theorem 3.8, $\text{Ext}(G) \subseteq S$. If $\text{Ext}(G) = V(G)$, then $S = V(G)$ and the proof is over. Suppose $\text{Ext}(G) \neq V(G)$. Let $X = V(G) - \text{Ext}(G)$. Then $X \neq \emptyset$. 


Let \( x \in X \). Let \( u, v \in \text{Ext}(G) \) such that \( uv \notin E(G) \). (This is possible since \( \text{Ext}(G) \neq V(G) \) and \( G \) is connected). Since \( u \) and \( v \) are connected by a geodesic path \( x \) has at least two neighbors in \( V(G) \). If \( xu \in E(G) \) or \( xv \in E(G) \), then \( u \) or \( v \) has exactly one neighbor \( x \) in \( V - S \) and so \( x \in S \). Now, assume \( xu \notin E(G) \) and \( xv \notin E(G) \). Suppose \( x \notin S \). Since \( S \) is a certified dominating set of \( G \), at least one element in \( N(v) \) belong to \( S \). If \( N(v) \subseteq S \), thus \( x \) has no neighbor in \( V - S \) and so \( S \) is not a geodetic certified dominating set of \( G \), which is a contradiction. Suppose \( N(v) \not\subseteq S \). Then there exists an element \( y \in N(v) \) such that \( y \notin S \). Since every element in \( N(v) \) lie on exactly one geodesic path and \( \text{Ext}(G) \neq V(G) \), \( G \) contains no cycle. This shows that there is a vertex \( z \) in \( S \), which has exactly one neighbor \( y \) in \( V - S \). It follows that \( S \) is not a geodetic certified dominating set of \( G \), again a contradiction. Hence \( x \in S \). Since \( x \in X \) if arbitrary, \( X \subseteq S \) and so \( S = X \cup \text{Ext}(G) \). Therefore, \( c_{\gamma_{g}}(G) = |S| = p \).

**Corollary 3.19.** For any tree \( T \), \( c_{\gamma_{g}}(T) = p \).

**Proof.** Since the set of all end vertices for a geodetic set of \( T \) and any two end vertices lie on a unique geodesic path, the result follows from Theorem 3.19.

**Open Problem.** Characterizes the graph for which \( c_{\gamma_{g}}(G) = p \)

**Theorem 3.20.** Let \( G \) be a connected triangle-free graph with \( f(G) \geq 2 \). If \( g(G) = 2 \), then \( c_{\gamma_{g}}(G) = \gamma_{\text{cer}}(G) \).

**Proof.** Let \( G \) be a connected triangle-free graph with \( f(G) \geq 2 \) and \( g(G) = 2 \). Let \( S \) be a minimum certified dominating set of \( G \) in which the distance between some vertices in \( S \) is as large as possible. Since \( g(G) = 2 \), there exists a pair of antipodal vertices \( x, y \in V(G) \) such that every vertices in \( G - \{x, y\} \) lies on the \( x - y \) geodesic path in \( G \). We show that \( S \) is a geodetic certified dominating set of \( G \), it is enough to show that \( S \) is a geodetic set of \( G \). We consider two cases.

Case 1. \( \{x, y\} \subseteq S \).

Then every vertices in \( G - S \) lies on the \( x - y \) geodesic path in \( G \), where \( x, y \in S \). Therefore \( S \) is a geodetic certified dominating set of \( G \).

Case 2. \( \{x, y\} \not\subseteq S \).
Then by the choice of $S$, we have three subcases.

Subcase 2a. $x \in S$ and $y \notin S$.

Since $f(G) \geq 2$, $y$ has at least two adjacent vertices, say $u$ and $v$. Since $G$ is triangle – free, that $uv \notin E(G)$. It follows that $y$ lies on the $u – v$ geodesic path in $G$. Since $S$ is a certified dominating set of $G$ and $y \notin S$, either $u$ or $v$ are both of them must be in $S$. If $u, v \in S$, then $S$ is a geodetic certified dominating set of $G$. If $u \in S$ and $v \notin S$, then there exists a vertex $u' \in S$ such that $v \in N(u')$. Since $G$ contains no triangle, that $x$ lies $u – u'$ geodesic path in $G$. Therefore $S$ is a geodetic set in $G$.

Subcase 2b. $x \notin S$ and $y \in S$.

As similar in subcase 2a, $S$ is a geodetic set in $G$.

Subcase 2c. $x \notin S$ and $y \notin S$.

Since $S$ is a certified dominating set of $G$, we have a element in $N(x) \in S$ and a element in $N(y) \in S$. If $I[S] = V(G)$, then $S$ is a geodetic set of $G$. Suppose $v \in V(G) – I[S]$. Since $S$ is a certified dominating set of $G$, a element $u \in N(v)$ such that $u \in S$. Since $G$ is triangle – free, $ux \notin E(G)$. If $x \in S$, then $v \in I[u, x] \subseteq I[S]$, which is a contradiction. Therefore $x \notin S$.

Also, $(G) \geq 2$, $x$ has neighbor $y$ in $S$ continuing this way we obtain that $g(G) > 2$, a contradiction. Hence, $I[S] = V(G)$ and so $S$ is a geodetic set of $G$.

In both the cases, $S$ is a geodetic certified dominating set of $G$ and $c\gamma_g(G) \leq |S| = \gamma_{cer}(G)$.

By Theorem 3.6, we conclude that $c\gamma_g(G) = \gamma_{cer}(G)$.

Theorem 3.21. For any two integer $3 \leq a \leq p$, $a \neq p – 1$, there exists a connected graph $G$ such that $c\gamma_g(G) = a$ and $|V(G)| = p$.

Proof. The Theorem can be easily verified for $3 \leq a \leq p$. Since if $p = 3$, then $G \in \{P_3, K_3\}$ and if $p = 4$, then $G = K_4$. Now we consider the case for $p \geq 5$. If $a = p$, then take $G = K_p$ or a tree. For $a \leq p – 2$, consider $\overline{K_2}$ with vertices $x$ and $y$. Now, add new vertices $y_1, y_2, \ldots, y_{p-a}$ and join each $y_i$ ($1 \leq i \leq p – a$) with $x$ and $y$. Also join new pendent vertex $x_1, x_2, \ldots, x_{a-2}$ with $y$ and thereby obtained a connected graph $G$ gives in Figure 3.5. Then the vertex set of $G$ as $V(G) = \{x, y, x_1, x_2, \ldots, x_{a-2}, y_1, y_2, \ldots, y_{p-a}\}$ and the set $S = \{x, y,$
\(x_1, x_2, \ldots, x_{a-2}\) is a minimum geodetic certified dominating set of \(G\). Hence \(|V(G)| = P\) and \(c_{\gamma_g}(G) = |S| = 2 + a - 2 = a\).

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