SOME REMARKS FOR IMPROVING COINCIDENT POINT THEOREMS IN BANACH SPACES

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ABSTRACT

In this paper we study on some remarks from improving coincident point theorems in Banach spaces. We observe that it is possible to improve every theorem involving four maps by weakening the condition of compatibility of two of the maps to that of weak compatibility. Finally, we point out that several papers involving rational contractive definitions have faulty proofs.

KEYWORD : Remarks, Improving, Coincident, Compatibility, Contractive

INTRODUCTION & CORE AREAS

Theorem 1. Let A, B, S, T be four selfmaps of a Banach space X such that

(1) $A(X) \subset T(X), B(X) \subset S(X)$, and for each x, y in X, and either

(2) $||A_x - By||) \leq \alpha \left\{ \frac{||Ax - By|| \;\; ||Sx - By|| + ||By - Ty|| \;\; ||Ty - Ax||}{||Sx - By|| + ||Ty - Ax} \right\} + \beta ||Sx - Ty||$ (1)

if $||Sx - By|| + ||Ty - Ax|| \neq 0$, where $\alpha, \beta > 0, \alpha + \beta < 1$, or $(2')$, $||Ax - By|| = 0$ if

$||Sx - By|| + ||Ty - Ax|| = 0$.

If either a [A, S] art compatible, A or S is continuous and (B, T) are weakly compatible, or (b) (B, T) are compatible, B or T is continuous, and (A, S) are weakly compatible, then A, B, S and T have a coincident point z. Moreover z is the coincident point of A and S and B and T.

PROOF : If $y_n = y_{n+1}$ for some n, then A, B, S and T have a coincident point.

In (1) set $x = x_{2n}. y = y_{2+1}$ to obtain

$$||y_{2n+1} - y_{2n+2}|| \le \alpha\left\{\frac{||y_{2n} - y_{2n+1}||\,||y_{2n} - y_{2n+1}|| + ||y_{2n+1} - y_{2n+2}||\,||y_{2n+1} - y_{2n+2}||}{||y_{2n} - y_{2n+1}|| + ||y_{2n+1} - y_{2n+2}||}\right\} + \beta||y_n - y_{2n+1}||$$

$$= \frac{\alpha||y_{2n} - y_{2n+1}||\,||y_{2n} - y_{2n+2}||}{||y_{2n} - y_{2n+2}||} + \beta||y_{n2} - y_{2n+1}||$$

Setting $x = x_{2n}, y = x_{2n-1}$ in (1) yields

$$(||y_{2n+1}, y_{2n}||) \le \alpha\left\{\frac{(||y_{2n} - y_{2n+1}||)\,(||y_{2n} - y_{2n}||) + (||y_{2n-1} - y_{2n}||)\,(||y_{2n-1} - y_{2n+1}||)}{(||y_{2n} - y_{2n}||) + (||y_{2n-1} - y_{2n+1}||)}\right\} + \beta||y_{2n}, y_{2n-1}||$$

$$= \frac{\alpha||y_{2n-1}, y_{2n}||\,||y_{2n-1} - y_{2n+1}||}{||y_{2n-1} - y_{2n+1}||} + \beta||y_{2n} - y_{2n-1}||$$

Therefore, for all n

$$||y - y|| \le \left\{\frac{\alpha||y_{n-1} - y_n||\,||y_{n-1} - y_{n+1}||}{||y_{n-1} - y_{n+1}||}\right\} + \beta||y_{n-1} - y_n|| \qquad (2)$$

Suppose that $y_n = y_{n+1}$ for some n. Then $y_n = y_{n+2}$. For, if not, then, from (2), with n replaced by n + 1, we get $||y_{n+1} - y_{n+2}|| = 0$, a contradiction. It then follows that $y_n = y_{n+k}, k = 1,2 \ldots$ Thus there exist points w_1 and w_2 such that $V_1 := Aw_1 = Sw_1$ and $V_2 := Bw_2 = Tw_2$. For, if not, then, by (1), $||Aw_1 - Bw_2|| \le \beta||Sw_1 - Tw_2||$, which implies that $Sw_1 = Bw_2$, a contradiction. Therefore, $v_1 = v_2$. Also it is the case that $Av_1 = Sv_1$ and $Bv_2 = Tv_2$. One must have $Sv_1 = Bv_2$ for, otherwise, from (2) one obtains $||Av_1 - Bv_2|| \le \beta||Sv_1 - Tv_1||$, which implies that $Sv_1 = Bv_2$. a contradiction. Therefore, $Sv_1 = Bv_2$. Since $v_1 = v_2$is a common coincidence point of A, B, S and T. Define $z = Av_1$. Then z is also a coincidence point of A, B, S and T. Moreover, $Sz = Bv_1$. For, otherwise, form (1), one obtains $||Az - Bv_1|| \le \beta||Sz - Tv_1||$, which implies that $Sz = Bv_1$. Thus $Sz = Sv_1 = z$, and z is a coincident point of A, B, S and T.

If $y_n = y_{n+2}$ for some n, then $||y_n - y_{n+2}|| + ||y_{n+1} - y_{n+1}|| = 0$, and, form (2), $||y_n - y_{n+1}|| = 0$; i.e., $y_n = y_{n+1}$, which we have already taken care of. Therefore, we may assume that $y_n \ne y_{n+2}$ for all n. From (2), one obtains

$$||y_n - y_{n+1}|| \le (\alpha + \beta)||y_{n-1} - y_n||,$$

is satisfied. Since conditions (a)-(b) of Lemma I have been taken care of, (e) holds. Let $z := \lim Sx_2$, and assume (a) and S continuous. Since A and S are compatible, $\lim Sax_{2n} = \lim ASx_{2n} = Sz$.

Suppose that $z \neq Sz$. Then

$\lim [||SSx_{2n} - Bx_{2x+1}|| + ||Tx_{2n+1} - Asx_{2n}||] = 2||Sz - z|| \neq 0$. Therefore, for all n sufficiently large (1) applies, and one obtains

$$||Asx_{2n} - Bx_{2n+1}|| \leq \alpha \max\{||ASx_{2n} - SSx_{2n}||, ||Bx_{2n+1} - Tx_{2n+1}||\} + \beta||SSx_{2n} - Tx_{2n-1}||$$

Taking the limit as $n \to \infty$ yields $||Sz - z|| \leq \beta||Sz - z||$, a contradiction. Therefore, $Sz = z$.

Suppose that $Az \neq z$. Then $\lim [||Sz - Bx_{2n+1}|| + ||Tx_{2n+1} - Az||] = ||z - Az|| \neq 0$.

Therefore, for all n sufficiently large, (1) applies to give

$$||Az - Bx_{2n+1}|| \leq \alpha \max\{||Az - Sz||, ||Bx_{2n+1} - Tx_{2n+1}||\} + \beta||Sz - Tx_{2n+1}||$$

Taking the limit as $n \to \infty$ gives $||Az - z|| \leq \alpha||Az - z||$, a contradiction. Therefore, Az = z.

Since $A(X) \subset T(X)$. there exists a $w \in X$ with $Az = Tw$. Suppose that $Bw \neq Tw$. Then (1) applies to give

$$||Az - Bw|| \leq \alpha \max\{||Az - Sz||), ||Bw - Tw||)\} + \beta||Sz - Tw||)\},$$

or, $||Tw - Bw|| \leq \alpha||Tw - Bw||$, a contradiction. Therefore, w is a coincident point for B ant T. Since they are weakly compatible, $Bz = BAz = BTw = TBw = TTw = Taz = Tz$, and z is also a coincident point for B and T.

Suppose that $z \neq Bz$. Then $\lim [||Sx_{2n} - Bz||) + ||Tz - Ax_{2n}||)] = ||zBz|| > 0$.

Therefore, for all n sufficiently large we can use (1) to obtain

$$||Ax_{2n} - Bz||) \leq \alpha \max\{||Ax_{2n} - Sx_{2n}||, ||Bz - Tz|| + \beta||Sx_{2n} - Tz||\}.$$

Taking the limit as $n \to \infty$ gives $||z - Bz|| \leq \beta||z - Bz||$, a contradiction. Therefore, $z = Bz = Tz$, and z is a coincident point of A, B, S and T.

The other parts are proved in a similar manner. The uniqueness conditions on z follow from (1).

Theorem 2. Let A, B, S, T be selfmaps of a Banach space X with $A(X) \subset T(X), B(X) \subset S(X)$ and satisfying, for each x, y in X either

$$||Ax - By|| \leq \frac{a||Sx - Ax||\ ||Ty - By|| + b||Sx, By||\ ||Ty - Ax||}{||Sx - Ax|| + ||Ty - By||} + c||Sx - Ty|| \qquad (3)$$

if $||Sx - Ax|| + ||By - Ty|| \neq 0$, $a \geq 0, 0 \leq c < 1, a + 2c < 2$, or

$$||Ax - By|| = 0 \text{ if } ||Sx - Ax||) + ||By - Ty|| = 0.$$

If either (a) (A, S} are compatible, A or S is continuous and (B, T) are weakly compatible, or (b), {B, T} are compatible, B or T is continuous, and (A, S) are weakly compatible, then A, B, S and T have a coincident point z in X. Further z is the coincident point of A and S and of B and T.

PROOF. In (3) set $x = x_{2n}, y = x_{2n+1}$ to get

$$||y_{2n+1} - y_{2n+1}|| \leq \frac{a||y_{2n} - y_{2n+1}||\ |||y_{2n+1} - y_{2n+2}||}{||y_{2n} - y_{2n+1}|| + ||y_{2n+1} - y_{2n+2}||} + c||y_{2n} - y_{2n+1}||$$

Setting $x = x_{2n+2}, y = x_{2n+1}$ in (3) yields

$$||y_{2n+2}, y_{2n+3}|| \leq \frac{a||y_{2n+2} - y_{2n+3}||\ |||y_{2n+1} - y_{2n+2}||}{||y_{2n+2} - y_{2n+3}|| + ||y_{2n+2} - y_{2n+1}||} + c||y_{2n+2}, y_{2n+1}||$$

Therefore, for all n,

$$||y_{n+1} - y_{n+2}|| \leq \frac{a||y_n - y_{n+1}||\ |||y_{n+1} - y_{n+2}||}{||y_n - y_{n+1}|| + ||y_{n+1} - y_{n+2}||} + c||y_n - y_{n+1}|| \qquad (4)$$

Suppose that there exists an n such that $y_n = y_{n+1}$. Then $y_{n+1} = y_{n+2}$. If not then it follows from (4) that $d(y_{n+1}, y_{n+2}) \leq 0$, a contradiction. Therefore, A and S and B and T have coincident points.

Assume that $y_n \neq y_{n+1}$ for each n. Then with $d_n := ||y_n - y_{n+1}||$, (4) implies $d_n d_{n+1} + d^2{}_{n+1} \leq a d_n d_{n+1} + c d^2{}_n + c d_n d_{n+1}$. The resulting inequality implies that

$$d_{n+1} \leq d_n \left[\frac{-(1-a-c) + \sqrt{(1-a-c)^2 + 4c}}{2}\right]$$

The quantity in brackets can be shown to be positive and less than one. Therefore, (1) is satisfied, and $\{y_n\}$ converges to a point z.

The remainder of the proof is similar to that of Theorem 1, and will therefore be omitted.

There are two contractive forms for three maps. One is obtained by setting T = S and the other is obtained by setting B = A. Also for three maps we can prove slightly more general results. For the situation in which T = S, set $x_0 \in X$ and define $\{x_n\}$ by $Ax_{2n} = Sx_{2n+1}, Bx_{2n+1} = Sx_{2n+2}, y_{2n} := Sx_{2n}, y_{2n+1} \coloneqq Sx_{2n+1}$.

Lemma 1. Let A, B, S be selfmaps of a Banach space X such that $A(X) \cup B(X) \subset S(X)$. Suppose there exists a $\lambda \in [0,1]$ such that

$$||y_n - y_{n+1}|| \leq \lambda ||y_{n+1} - y_n|| \text{ for } y_n \neq y_{n+1} \tag{5}$$

Assume that $\{y_n\}$ is complete

Then, either

(a) A and S have a coincidence point,

(b) B and S have a coincidence point, or

(c) $\{y_n\}$ converges to a point $z \in X$ and

PROOF. Suppose there exists an n for which $y_{2n} = y_{2n+2}$ Then $Sx_{2n} = Sx_{2n+1} = Ax_{2n}$, and (a) holds. If $y_{2n+1} = y_{2n+2}$ for some n then a similar argument leads to condition (b).

Assume that $y_n \neq y_{n+1}$ for each n.

Theorem 3. Let A, B, S be three selfmaps of a complete metric space (X, d) such that, for all x, y in X, either

$$||Ax - By|| \leq \frac{a||Sx - Ax||\,||Sy - By|| + b||Sx - By||\,||Sy - Ax||}{||Sx - Ax|| + ||Sy - By||}$$

$$+c\frac{||Sx - Ax||\,||Sy - Ax|| + ||Sy - By||\,||Sx - By||}{||Sx - By|| + ||Sy - Ax||} \tag{6}$$

if $||Sx - Ax|| + ||Sy - By|| \neq 0$ and $||Sx - By|| + ||Sy - Ax|| \neq 0$, where $a, b, c \geq 0$ with $a + 2c < 2$, or

$||Ax - By|| = 0$ if $||Sx - Ax|| + ||Sy - ||By|| = 0$ or $||Sx - By|| + ||Sy - Ax|| = 0$ (7)

If $A(X) \cup B(X) \subset S(X)$ and, if either S is continuous and is compatible with either A or B, or if A is continuous and is compatible with S or B is continuous and is compatible with S, then A, B and S have a coincident point z. Further, z is the only coincident point of A and S and B and S.

PROOF. Suppose that $y_{2n} = y_{2n+1}$ for some n. In (15) set $x = x_{2n}, y = x_{2n+1}$ to get

$$||y_{2n+1} - y_{2n+2}|| \leq \frac{c||y_{2n+1} - y_{2n+2}|| \, ||y_{2n} - y_{2n+2}||}{||y_{2n} - y_{2n+2}||}$$

If $y_{2n+1} \neq y_{2n+2}$, then it follows from the above inequality that $||y_{2n+1} - y_{2n+2}|| \leq c||y_{2n+1} - y_{2n+2}||$, a contradiction, since c < 1. Therefore, $y_{2n+1} = y_{2n+2}$ If there exists an n such that $y_{2n-1} = y_{2n}$, then if follows that $y_{2n+1} = y_{2n+1}$Therefore, if there exists an for which $y_n = y_{n+1}$, then one has $y_n = y_{n+k}$ where k = 1, 2,…

Thus there exist two points $w_1, w_2 \in X$ such that $v_1 := Aw_1 = Sw_1$ and $v_2 := Bw_2 = Sw_2$. Since $||Sw_1 - Aw_1|| + ||Sw_2 - Bw_2|| = 0$, it follows from (7) that $||Aw_1 - Bw_2|| = 0$; i.e., $v_1 = v_2$.

Suppose that A and S are compatible. Then they commute at coincidence points, and $v_1 = Aw_1 = Sw_1$ implies that $Av_1 = Sv_1 = Sv_1$, Since $||Sw_1 - Aw_1|| + ||Sw_2 - Bw_2|| = 0$, it follows from (7) that $||Av_1 - Bw_2|| = 0$; i.e., $Av_1 = Bw_2$. But $Bw_2 = v_2 = v_1$.

Therefore, $v_1 = Av_1, = Sv_1$, Since $||Sv_1 - Av_1|| + ||Sv_2 - Bv_2|| = 0$, it follows from (7) that $||Av_1 - Bv_1|| = 0$; i.e., $Av_1 = Bv_2 = Bv_1$ and $v_1$is a common fixed point of A, B, and S. The other possibilities are handled in a similar manner.

Now assume that $y_n \neq y_{n+1}$ for each n, then it can be shown, using (6), that (5) is satisfied with $\lambda = (2a + c - 2)/(2 - c) < 1$. From Lemma 2$\{y_n\}$ converges. Call the limit z.

Suppose that S is continuous and is compatible with A. Then lim $ASx_{2n} = \lim SAx_{2n} = Sz$.

Suppose that Sz ≠ Bz. Then, in (6) set $x = Sx_{2n}, y = z$ to obtain

$$||ASx_{2n} - Bz|| \leq \frac{a||SSx_{2n} - Asx_{2n}||\ ||Sz - Bz||) + b||SSx_{2n}, Bz||\ ||Sz - ASx_{2n}||}{||SSx_{2n} - Asx_{2n}|| + (||Sz - Bz||)} +$$

$$c\frac{||SSx_{2n} - Asx_{2n}||\ ||Sz - Bz||)\ ||Sz - ASx_{2n}|| + ||Sz - Bz||\ ||SSx_{2n}, Bz||}{||(SSx_{2n} - Bz)|| + ||Sz - ASx_{2n}||}$$

Taking the limit as n → ∞ yields

$$||Sz - Bz|| \leq \frac{0+0}{||Sz - Az||} + \frac{0 + c[||Sz - Bz||]^2}{||Sz - Bz||} = c||Sz - Bz||$$

a contradiction. Therefore, Sz = Bz.

Suppose that Az ≠ Bz = Sz. Then, from (6) with x = y = z.

$$||Az - Bz|| \leq \frac{\alpha||Sz - Az||\ ||Sz - Az|| + b||Sz - Bz||\ ||Sz - Az||}{||Sz - Az|| + ||Sz - Bz||}$$

$$+c\frac{||Sz - Az||\ ||Sz - Az|| + ||Sz - Bz||\ ||Sz - Bz||}{||Sz - Bz|| + ||Sz - Az||}$$

$$= \frac{0+0}{||Sz - Az||} + \frac{[||Sz - Az||]^2}{||Sz - Az||} - c||Sz - Az|| = c||Bz - Az||$$

a contradiction. Therefore, Az = Sz = Bz.

Since A and S are compatible, they commute at coincidence points. Thus AAz = ASz = SAz = SSz. Since $||SS - AS|| + ||Sz - Bz|| = 0$, we have, from (7) that ASz = Bz. But Bz = Sz. Therefore, Sz is a non-invariant point of A.

Also SSz = SAz = Bz and Sz is a non-invariant point of S.

Suppose that Sz ≠ BSz, then, from (6),

$$||Az - BSz|| \leq \frac{\alpha||Sz - Az||\ ||SSz - BSz||\ + b||Sz - BSz||\ ||SSz, Az||}{||Sz - Az|| + ||SSz - BSz||}$$

$$+c\ \frac{||Sz-Az||\ ||SSz-Az||+||SSz-BSz||\ ||Sz-BSz||}{||Sz-BSz||+||SSz-Az||}$$

$$= \frac{0+0}{||Sz - BSz||} + \frac{c[||Sz - BSz||]^2}{||Sz - BSz||} = c||Sz - BSz|| = c||Az - BSz||$$

a contradiction. Therefore, Sz is a coincident point of A, B and S.

Similarly, if S is continuous and is compatible with B, one obtains the same conclusion.

Suppose that A is continuous and compatible with S. Then $\lim ASx_{2n} = \lim Sax_{2n} = Az$. Since A(X) ⊂ S(X) there exists a point w ∈ X such that Az = Sw.

Suppose that Bw ≠ Az. Then, from (6) with $x = Ax_{2n}, y = w$, one obtains

$$||AAx_{2n}, w|| \leq \frac{\alpha||SAx_{2n} - AAx_{2n}||\ ||Sw - Bw||\ + b||SAx_{2n} - Bw||\ ||Sw - AAx_{2n}||)}{||SAx_{2n} - AAx_{2n}|| + ||Sw - Bw||}$$
$$+c\frac{||SAx_{2n} - AAx_{2n}||\ ||Sw - AAx_{2n}|| + ||Sw - Bw||\ ||SAx_{2n} - Bw||}{||SAx_{2n} - Bw|| + ||Sw - AAx_{2n}||}$$

Taking the limit as n → ∞ one obtains

$$||Az, Bw|| \leq \frac{0+0}{||Sw - Bw||} + \frac{c||Sw - Bw||\ ||Az - Bw||)}{||Az - Bw||} - c||Az - Bw||$$

a contradiction. Therefore Az = Bw = Sw.

Suppose that Aw ≠ Bw. Then from (6) with x = y = w,

$$||Aw - Bw|| \leq \frac{0+0}{||Sw - Aw||} + \frac{c[||Sw - Aw||]^2 + 0}{||Sw - Aw||} = c||Sw - Aw|| = c||Bw - Aw||,$$

a contradiction. Therefore, Aw = Bw = Sw. Since A and S are compatible, AAW = ASw = Saw = SSw. Since $||SAw - AAw|| + ||Sw - Bw|| = 0$, we have, from (7) that AAw = Bw.

Since Bw = Aw, Aw = Sw = Bw is a coincident point of A. SSw = Saw = ASw = AAw = Bw = Sw and Sw is a coincident point of S.

Suppose that BSw ≠ Sw. Then from (6) with x = w, y = Sw,

$$||Aw - BSw|| \leq \frac{\alpha||Sw - Aw||\ ||SSw - BSw|| + b||Sw - BSw||\ ||SSw - Aw||}{||Sw - Aw|| + ||SSw - BSw||}$$

$$+c\frac{||Sw - Aw||\ ||SSw - Aw||) + ||SSw - BSw||\ ||Sw - BSw||}{||Sw - BSw|| + ||SSw - Aw||}$$

$$= \frac{0 + 0}{||Sw - BSw||} + \frac{c[||Sw - BSw||]^2}{||Sw - BSw||} = c||Sw - BSw|| = c||Aw - BSw||,$$

a contradiction.

CONCULTION

Therefore, Sw is a coincident point of A, B and S. The proof for B compatible with S is similar.

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