IMPROVING COINCIDENT POINT THEOREMS FOR THREE MAPS IN BANACH SPACES

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ABSTRACT

In this paper we study on improving coincident point theorems for three maps in Banach spaces. We also point out that a number of coincident point theorems in the literature, for three maps, involving rational contractive conditions. We observe that it is possible to improve every theorem involving three maps.

KEYWORDS : Improving, Commuting, Contradiction, Uniqueness.

INTRODUCTION & CORE AREAS

Theorem 1. Let E, F, and T be three commuting selfmaps of a Banach space X such that one of the maps is continuous and E, F, T satisfy :

(i) $E(X) \cup F(X) \subset T(X)$.

(ii) there exist positive integers m, n such that

$$||E^m x - F^n y|| \leq \frac{\alpha||Tx - E^m x|| \, ||Ty - F^n y||)}{||Tx - F^n y|| + ||Ty - E^m x|| + ||Tx - Ty||)} \beta||Tx - Ty|| \quad (1)$$

for each x, y in X such that $||Tx - F^n y|| + ||Ty - E^m x|| + ||Tx - Ty|| \neq 0$, where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$ and

$$||E^m x - F^n y|| = 0 \text{ whenever } ||Tx - F^n y|| + ||Ty - E^m x|| + ||Tx - Ty|| = 0 \quad (2)$$

Then E, F and T have a unique coincident point z. Moreover, z is the only common fixed point of E and T and F and T.

PROOF. Define $\{y_n\}$ by $y_{2n+1} = E^m x_{2n}, y_{2n+2} = F^n x_{2n+1}$. Then, from (1),

$$||E^m x_{2n} - F^n x_{2n+1}|| \leq \frac{\alpha||Tx_{2n}, E^m x_{2n}||\, ||Tx_{2n}, F^n x_{2n+1}||}{||Tx_{2n} - F^n x_{2n+1}|| + ||Tx_{2n+1} - E^m x_{2n}|| + ||Tx_{2n} - Tx_{2n+1}||} + \beta||Tx_{2n} - Tx^n{}_{2n+1}||$$

$$||y_{2n+1} - y_{2n+2}|| \leq \frac{\alpha||y_{2n} - y_{2n+1}||\, ||y_{2n+1}, y_{2n+2}||}{||y_{2n} - y_{2n+2}|| + 0 + ||y_{2n} - y_{2n+1}||} + \beta||y_{2n} - y_{2n+1}||$$

or,

Using (1) with $x = x_{2n+2}, y = x_{2n+1}$

$$||y_{2n+3} - y_{2n+2}|| \leq \frac{\alpha||y_{n+2} - y_{2n+3}||\, ||y_{2n+1}, y_{2n+2}||}{0 + ||y_{2n+1}, y_{2n+3}|| + ||y_{2n}, y_{2n+1}||} + \beta||y_{2n+2} - y_{2n+1}||$$

Thus, for each n,

$$||y_{n+1} - y_{n+2}|| \leq \frac{\alpha||y_{n+1} - y_{n+2}||\, ||y_n - y_{n+1}||}{||y_n - y_{n+1}|| + ||y_n - y_{n+2}||} + \beta||y_n - y_{n+1}|| \quad (3)$$

Suppose that $y_n = y_{n+1}$ for some n. Then one must have also that $y_{n+1} = y_{n+2}$. If not, then from (3) $y_{n+1} = y_{n+2}$, a contradiction. Therefore $y_{n+1} = y_{n+2}$ and, in general, $y_n = y_{n+1}$ for k = 1, 2, … . Thus there exist points w_1 and w_2 in X such that $Tw_1 = E^m w_1$and $Tw_2 = F^n w_2$ that $y_n = y_{n+1}$ for some n. Then one must have also that $y_{n+1} = y_{n+2}$, if not, then from (3) $y_{n+1} = y_{n+2}$, contradiction. Therefore, $y_{n+1} = y_{n+2}$, and, in general, $y_n = y_{n+k}$ for k = 1, 2, .. . Thus there exist points w_1 and w_2 in X such that $Tw_1 = E^m w_1$and $Tw_2 = F^n w_2$.

Claim $-$ $Tw_1 = Tw_2$ suppose not. Then $||Tw_1 - Tw_2|| \neq 0$ and (1) applies to give

$$||E^m w_1 - F^n w_2|| \leq \frac{\alpha||Tw_1 - E^m w_1||\, ||Tw_2 - F^n w_2||}{||Tw_1 - F^n w_2|| + ||Tw_2 - E^n w_1|| + ||Tw_1 - Tw_2||}$$

$$+\beta||Tw_1 - Tw_2|| = \beta||Tw_1 - Tw_2||$$

a contradiction. Therefore, $Tw_1 = Tw_2$. Thus there exists a point $z = Tw_1 - E^m w_1 = Tw_2 = F^n w_2$. Since E^m and T commute, $Tz = TE^m w_1 = E^m Tw_1 = E^m z$. Also, $Tz = TF^n w_2 = F^n Tw_2 = F^n z$. and z is a coincident point of T, E^m and F^n.

Claim $Tw_1 = E^m z$. Suppose not Then (1) applies to give

$$||E^m z - F^n w_2|| \leq \frac{\alpha||Tz - E^m z|| \, ||Tw_2 - F^n w_2||}{||Tz - F^n w_2|| + ||Tw_2 - E^m z|| + ||Tz - Tw_2||} = 0$$

a contradiction.

Therefore $E^m z = F^n w_2 = Tw_2 = Tw_1 = z$, and z is a coincident point of T, E^m and F^n

Assume that $y_n \neq y_{n+1}$ for each n. Then it follows from (1) that

$$||y_{n+1} - y_{n+2}|| \leq (\alpha + \beta)||y_n - y_{n+1}||,$$

Let $z = \lim y_n$.

Assume that T is continuous. Then, since E and T commute, $\lim E^m Tx_{2n} = T$

$E^m x_n = Tz$.

Suppose that $z \neq Tz$ Then

$$\lim[||TTx_{2n} - F^n x_{2n+1}||) + ||Tx_{2n+1} - E^m Tx_{2n}||) + ||TTx_{2n} - Tx_{2n+1}|| = e||Tz - z|| > 0.$$

Therefore, for all n sufficiently large (1) applies. Note that $||Tx - F^n y|| + ||Tx - Ty|| \geq ||Ty - F^n y||$. Therefore (1) implies that, for all n sufficiently large.

$$||E^m Tx_{2n} - F^n x_{2n+1}|| \leq \alpha(||TTx_{2n} - E^m Tx_{2n})|| + \beta||TTx_{2n} - Tx_{2n+1}||.$$

Taking the limit is $n \to \propto$ gives $||Tz - z|| \leq \beta||Tz - z||$, a contradiction. Therefore $z = Tz$.

Suppose that $E^m z \neq z$. Then

$$\lim[||Tz - F^n x_{2n+1}|| + ||Tx_{2n+1} - E^m z|| + ||Tz - Tx_{2n+1}||] = ||(z - E^m z)|| > 0.$$

Therefore, for all n sufficiently large, (1) applies, and we obtain

$$||E^m z - F^n x_{2n+1}|| \leq \alpha||Tz - E^m z|| + \beta||Tz - Tx_{2n+1}||.$$

Taking the limit as $n \to \propto$ yields $||E^m - z|| \leq a||z - E^m z||$, a contradiction. Therefore, $z = E^m z$.

Claim $z = F^n z$.

If not, then $||Tz - F^n z|| + ||Tz - E^m z|| + ||Tz - Tz|| = ||z - F^n z|| > 0$ and (1) applies to give

$$||E^m z - F^n z|| \leq \alpha||Tz - E^m z|| + \beta||Tz - Tz|| = 0,$$

a contradiction. Therefore, z is a coincident point of E^m, F^n and T.

The other cases are proved in a similar manner. Thus in all cases one obtains a common fixed point. Condition (1) implies that z is the unique coincident point of T, E^m and F^n, and that it is the only coincident point T and E^m and T and F^n.

Consequently $E^m(Ez) = E(E^m z) = Ez$ and Ez is also a non-invariant point of E^m. Since E and T commute, from z = Tz, Ez = ETz = TEz, and Ez is also a coincident point of E^m and T. The second property of z implies that Ez = z. In a similar manner it can be shown that Fz is also a coincident point of F^n and T. from which it follows that Fz = z. Therefore, z is a coincident point of E, F and T. Condition (1) forces the uniqueness properties of z.

Theorem 2. Let A, B, and S be three selfmaps of a Banach space X satisfying

(i) $A(X) \cup B(X) \subset S(X)$,

(ii) for all $x, y \in X$ such that $||Sx, Sy|| + ||Sy, By|| \neq 0$,

$$||Ax - By|| \leq \alpha_1 \left[\frac{||Sx - By||\ ||Sx - Sy||}{||Sx - Sy|| + ||Sy - By||}\right] + \alpha_2[||Sx - Ax|| + ||Sy, By||]$$

$$+\alpha_3[||Sx - By|| + ||Sy - Ax||] + \alpha_3[||Sx - Sy||. \qquad (4)$$

Where $\alpha_1 \geq 0, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$ and

(iii) if $||Sx - Sy|| + ||Sy - By|| = 0$, then $||Ax - By|| = 0$.

If either

(iv) $\{A, S\}$ are compatible and A or S is continuous, or

(v) (B, S) are compatible and B or S is continuous, then A, B and S have a coincident point, which is also the coincident point of A and S and B and S.

PROOF. In (4) set $x = x_{2n}, y = x_{2n+1}$ to get

$$||y_{2n+1} - y_{2n+2}|| \leq \alpha_1 \left[\frac{||y_{2n} - y_{2n+2}||\,||y_{2n} - y_{2n+1}||}{||y_{2n} - y_{2n+1}|| + ||y_{2n+1} - y_{2n+2}||}\right]$$

$$+\alpha_2[||y_{2n} - y_{2n+1}|| + ||y_{2n+1} - y_{2n+2}||] + \alpha_3[||y_{2n} - y_{2n+2}|| + 0] + \alpha_4||y_{2n} - y_{2n+1}|| \quad (5)$$

Suppose that there exists an n such that $y_{2n} = y_{2n+1}$. It then follows that $y_{2n+1} = y_{2n+2}$. For if not, from (5) one obtains

$$||y_{2n+1} - y_{2n+2}|| \leq \alpha_2||y_{2n+1} - y_{2n+2}|| + \alpha_3[||y_{2n} - y_{2n+1}|| + ||y_{2n+1} - y_{2n+2}||$$

$$= (\alpha + \alpha)||y_{2n+1} - y_{2n+2}||.$$

which implies that y_{2n+1}, y_{2n+2}.

In (4), set $x = x_{2n+2}, y = x_{2n+1}$, to obtain

$$||y_{2n+3} - y_{2n+2}|| \leq \alpha_1 \left[\frac{||y_{2n+3}-y_{2n+2}||\,||y_{2n+2}-y_{2n+1}||}{||y_{2n+2}-y_{2n+1}||+||y_{2n+1}-y_{2n+2}||}\right] + \alpha_2[||y_{2n+2} - y_{2n+3}||$$

$$+||y_{2n+1} - y_{2n+2}||] + \alpha_3[0 + ||y_{2n+1} - y_{2n+3}||] + \alpha_4||y_{2n+2} - y_{2n+1}||] \quad (6)$$

Suppose there exist an n such that $y_{2n+1} = y_{2n+2}$. Then $y_{2n+2} = y_{2n+3}$. For otherwise, from (6) one obtains

$$||y_{2n+3} - y_{2n+2}|| \leq (\alpha_2 + \alpha_3)||y_{2n+3} - y_{2n+2}||. \quad (7)$$

or, $y_{2n+2} = y_{2n+3}$ a contradiction. Therefore, $y_{2n+2} = y_{2n+3}$. It then follows that $y_{2n+3} = y_{2n+4}$. If not then, using (5), one obtains $y_{2n+3} = y_{2n+4}$ a contradiction.

Therefore, if there exists an n such that $y_n = y_{n+1}$, it follows that $y_n = y_{n+kc}$ for k = 1, 2 ….. Thus there exist points $w_1, w_2 \in X$ such that $Aw_1 = Sw_1$ and $Bw_2 = Sw_2$. A, B and S have a coincident point.

From Lemma 2$\{y, .\}$ converges. Call the limit z.

Suppose that A is continuous. Since A and S are compatible, $\lim ASx_{2n} - \lim Ax_{2n} = Sz$.

Suppose that $z \neq Sz$, then

$$\lim [||SSx_{2n} - Sx_{2n+1})|| + ||Sx_{2n+1} - Bx_{2n+1}||] = ||Sz - z|| > 0$$

Therefore, for all n sufficiently large, (4) applies and one obtains

$$(||Sx_{2n} - Bx_{2n+1}||) \leq \alpha_1 ||SSx_{2n} - Sx_{2n+1}|| + \alpha_2 [||SSx_{2n} - Ax_{2n}|| - ||Sx_{2n+1} - Bx_{2n+1}||]$$

$$+\alpha_3 [||SSx_{2n} - Bx_{2n+1}|| + ||Sx_{2n+1} - ASx_{2n}||] + \alpha_4 ||SSx_{2n} - Sx_{2n+1}||$$

Taking the limit as $n \to \infty$ yields $||(Sz - z)|| \leq (\alpha_1 + 2\alpha_3 + \alpha_4)||Sz - z||$. a contradiction. Therefore z = Sz.

Suppose that $z \neq Az$. Then, since $y_n \neq y_{n+1}$ for all n, (4) implies and one obtains

$$||(Az - Bx_{2n+1})|| \leq \alpha_1 ||Sz - Sx_{2n+1})|| + \alpha_2 [||(Sz - Az)|| + ||Sx_{2n+1} - Bx_{2n+1}||]$$

$$+\alpha_3 [||Sz - Bx_{2n+1}|| + ||Sx_{2n+1} - Az||] + \alpha_4 [||Sz - Sx_{2n+1}|| \quad (8)$$

Taking the limit as $n \to \infty$ yields $||Az - z|| \leq (\alpha_2 + \alpha_3)||z - Az||$, a contradiction. Therefore, z = Az.

Suppose that $z \neq Bz$. Then $||Sz - Sz|| + ||Sz - Bz|| = ||z - Bz|| > 0$ and (4) applies to give

$$||Az - Bz|| \leq \alpha_1 ||Sz - Sz|| + \alpha_2 [||Sz - Az|| + ||Sz - Bz||]$$

$$+\alpha_3 [||Sz - Bz|| + ||Sz - Az||] + \alpha_4 [||Sz - Sz||,$$

or, $||z - Bz|| \leq (\alpha_2 + \alpha_3)||z - Bz||$, a contradiction. Therefore, z = Bz = z is a coincident point of A, B and S.

The other parts are proved similarly. Condition (20) implies the uniqueness conditions.

Corollary 1. Let A, S and T be selfmaps of a Banach space X such that

(a) $\{A, S\}$ and $\{A, T\}$ are compatible, $A(X) \subset S(X)$ and $A(X) \subset T(X)$ and,

(b) there exists a number q, $0 \leq 4 < 1$ such that, for each $x, y \in X$.

$$||Ax - Ay|| \geq q \max \left\{||Sx - Ty||, ||Sx - Ax||\, ||Ty - Ay||, \frac{||Sx - Ay|| + ||Ty - Ax||}{2}\right\}, \quad (9)$$

and

(c) S is continuous and is compatible with A, or T is continuous and is compatible with A, or A is continuous and is compatible with S and T.

Then A, S, and T have a coincident point z. Moreover, z is the coincident point common to A and S and A and T.

CONCLUSION

In the above study it has been tried to furnish the important results in this area and further scope for the study in future.

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