# Some Application of Semi-continuous function in topological space

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**Abstract :** In this paper, we introduce a function in the topological space, namely Semi-continuous function in the topological space. We find characterizations of these functions. Further, we study some fundamental properties of Semi-continuous function in the topological space.

**Keywords :** Open set, Closed set, Interior of a set, Closure of a set & Semi-continuous function.

## I. Introduction

Consequent upon the introduction of Semi-open sets, the continuity concept was generalized to the concept of Semi-continuity by the mathematician Norman Levine [1] in the year 1963. Semi-continuity due to Norman Levine [1] is one of the most important weak forms of continuity in the topological space. Then after, the mathematician S. G. Crossley & S. K. Hildebrand [4] defined irresolute functions by utilizing Semi-open sets due to Norman Levine [1]. These mathematician in turn generalized the concept of continuous functions to irresolute functions in the topological spaces.

In this paper, analogous to Norman Levine's[1] Semi-continuity, we investigate the certain results of Semi-continuity in the topological space.

## II. Preliminaries

Throughout the paper $(X , \tau)$ or simply X denote topological space on which no separation axioms are assumed unless otherwise mentioned explicitly. Throughout this paper, $X_1$ and $X_2$ etc are topological spaces then the function $f : X_1 \longrightarrow X_2$ is a single valued function from $X_1$ into $X_2$ . When A is a subset of $(X , \tau)$ then $C_L(A)$ & $I_N(A)$ are denote the closure and interior of the set A in the topological space.

**2.1 . <u>Semi-continuous function at a point</u> :** A function $f$ of a topological space $(X_1 , T_1)$ into a topological space $(X_2 , T_2)$ is said to be **Semi-continuous function at a point $x \in X_1$** if for each open set G of $f(x)$ in $X_2$, $\exists$ a Semi-open set H of 'x' in $X_1$ such that $f(H) \subseteq G$ or $H \subseteq f^{-1}(G)$.

**Or, A function $f : (X_1 , T_1) \longrightarrow (X_2 , T_2)$ is said to be Semi-continuous if and only if it is Semi-continuous at each point of the set $X_1$.**

**2.2 . <u>Semi-continuous function in terms of open sets</u> :** A function $f$ of a topological space $(X_1 , T_1)$ into a topological space $(X_2 , T_2)$ is said to be **Semi-continuous** is that for each open set H in $X_2$, $f^{-1}(H)$ is Semi-open set in $X_1$ .

**Or, A function $f$ is said to be Semi-continuous if and only if the inverse image of any open set is Semi-open set.**

**2.3 . <u>Semi-continuous function in terms of closed sets</u> :** A function $f$ of a topological space $(X_1 , T_1)$ into a topological space $(X_2 , T_2)$ is said to be **Semi-continuous** is that for each closed set F in $X_2$, $f^{-1}(F)$ is Semi-closed set in $X_1$.

**Or, A function $f$ is said to be Semi-continuous if and only if the inverse image of any closed set is Semi-closed set.**

**Clearly, every continuous function $f : (X_1, T_1) \longrightarrow (X_2 , T_2)$ is Semi-continuous but converse is however false that is a Semi-continuous function $f$ may fail to be continuous.**

We show this by an example.

Let $X = Y = [0 , 1]$ and consider a function $f : X \longrightarrow Y$ defined as

$f(x) = 1 , \; 0 \leq x \leq \frac{1}{2}$ and

$f(x) = 0 , \; \frac{1}{2} < x \leq 1$ .

This shows that the function $f$ is not continuous on [0 , 1].

But, the function $f$ is Semi-continuous on [0 , 1].

Since, the function $f$ takes only two values 1 or 0.

The inverse image of an open subset of Y is either φ or $[0, \frac{1}{2}]$ or $]\frac{1}{2}, 1]$ or X.

**Also, each of these sets are Semi-open set in X.**

**2.4 . <u>Theorem</u> : Let $(X_1, T_1)$ and $(X_2, T_2)$ be two topological spaces then a necessary & sufficient condition for a function $f : (X_1, T_1) \longrightarrow (X_2, T_2)$ to be Semi-continuous at a point $a \in X_1$ is that for each neighbourhood H of $f$(a) in $X_2$, $f^{-1}$(H) is a Semi-neighbourhood of the point 'a'.**

**<u>Proof</u> :**

**<u>Necessary</u> :** Let the function $f$ is Semi-continuous at a point 'a' ∈ X and let G be any neighbourhood of $f$(a) . So, a Semi-neighbourhood H of the point 'a' such that $H \subseteq f^{-1}(G)$ and $f^{-1}(G)$ contains a Semi-neighbourhood of the point 'a'.

**Hence, $f^{-1}$(G) is itself a neighbourhood of the point 'a'.**

**<u>Sufficient</u> :** Let for each neighbourhood G of $f$(a) then $f^{-1}$ (G) is a Semi-neighbourhood of the point 'a'. We put H = $f^{-1}$(G) then H is a Semi-neighbourhood of the point 'a' & $f(H) = f\{ f^{-1}(G)\} \subseteq G$. **Hence, the function $f$ is Semi-continuous at a point a ∈ X .**

**2.5 . <u>Theorem</u> : Let $(X_1, T_1)$ and $(X_2, T_2)$ be two topological spaces then a necessary & sufficient condition for a function $f : (X_1, T_1) \longrightarrow (X_2, T_2)$ to be Semi-continuous at a point 'a' $\in X_1$ is that for each open set G containing $f$(a) in $X_2$ , ∃ a Semi-open set H containing the point 'a' such that $f(H) \subseteq G$.**

**<u>Proof</u> :**

**<u>Necessary</u> :** Let the function $f$ is Semi-continuous at a point 'a' $\in X_1$ and let G be any open set containing $f$(a) in $X_2$ then G is a nbd. of $f$(a) so ∃ a Semi-nbd. U containing the point 'a' such that $f(U) \subseteq G$. So, ∃ a Semi-open set H containing the point 'a' such that $H \subseteq U$. Therefore, $f(H) \subseteq f(U) \subseteq G$. So, $f(H) \subseteq G$.

**<u>Sufficient</u> :** Let the given condition is satisfied.

Let G be any nbd. of $f$(a) so ∃ an open set O containing $f$(a) such that $O \subseteq G$. But, ∃ a Semi-open set H containing the point 'a' such that $f(H) \subseteq O \subseteq G$.

**Hence, the function $f$ is Semi-continuous at a point 'a' $\in X_1$ .**

**2.6 . Theorem : Let $(X_1, J_1)$ and $(X_2, J_2)$ be two topological spaces then a necessary & sufficient condition for a function $f : (X_1, J_1) \longrightarrow (X_2, J_2)$ to be Semi-continuous is that for each open set H in $X_2$, $f^{-1}$ (H) is a Semi-open set in $X_1$.**

**Or, the function $f$ is Semi-continuous if and only if the inverse image of every open set is a Semi-open set.**

**Proof :**

**Necessary :** Let the function $f$ is Semi-continuous and let G be any open set in $X_2$.

If $f^{-1}(G) = \varphi$, then $f^{-1}$ (G) is Semi-open set in $X_1$.

If $f^{-1}(G) \neq \varphi$, so let 'a' $\in f^{-1}(G)$ be arbitrary so $f(a) \in G$.

Thus, G is an open set containing $f(a)$ so G is a nbd. of $f(a)$.

Hence, $f^{-1}(G)$ is a Semi-nbd. of the point 'a'. Thus, $f^{-1}(G)$ is a Semi-nbd. of each of its points.

**Hence, $f^{-1}$ (G) is a Semi-open set in $X_1$.**

**Sufficient :** Let $f^{-1}(G)$ is a Semi-open set in $X_1$ for every open set G in $X_2$.

Let 'a' be an arbitrary point of set $X_1$ then we have to prove that the function $f$ is Semi-continuous at a point 'a'. Let V be any open set containing $f(a)$ so $f^{-1}(V)$ is a Semi-open set containing the point 'a'. Put $U = f^{-1}(V)$ then U is a Semi-open set containing the point 'a'.

Moreover, $f(U) = f\{ f^{-1}(V)\} \subseteq V \Longrightarrow f(U) \subseteq V$.

**Hence, the function $f$ is Semi-continuous at a point 'a'.**

**Since, the function $f$ is Semi-continuous at an arbitrary point 'a' of the set $X_1$.**

**So, the function $f$ is Semi-continuous over $X_1$ .**

**2.7 . Theorem : Let $(X_1, J_1)$ and $(X_2, J_2)$ be two topological spaces then a necessary & sufficient condition for a function $f : (X_1, J_1) \longrightarrow (X_2, J_2)$ to be Semi-continuous is that for each closed set F in $X_2$, $f^{-1}$ (F) is a Semi-closed set in $X_1$ .**

**Or, the function $f$ is Semi-continuous if and only if the inverse image of any closed set is a Semi-closed set.**

**Proof :** Since, the set F is closed set if and only if $F^c$ is an open set.

**Necessary :** Let the function $f$ is Semi-continuous and let F be any closed set in $X_2$ then $F^c$ is an open set in $X_2$ so $f^{-1}(F^c)$ is a Semi-open set in $X_1$. But, $f^{-1}(F^c) = \{f^{-1}(F)\}^c$ .

Hence, $\{f^{-1}(F)\}^c$ is a Semi-open set in $X_1$.

**Thus, $f^{-1}$ (F) is a Semi-closed set in $X_1$.**

**Sufficient** : Let for every closed set F in $X_2$, $f^{-1}(F)$ is a Semi-closed set in $X_1$.

Let G be any open set in $X_2$ then $f^{-1}(G^c) = \{f^{-1}(G)\}^c$ is a Semi-closed set in $X_1$.

So, $f^{-1}(G)$ is a Semi-open set in $X_1$.

**Hence, the function $f$ is Semi-continuous.**

**2.8 . Proposition : Let $J_1$ and $J_2$ be two topologies on a set X then the identity function $I : (X , J_1) \longrightarrow (X , J_2)$ is defined by $I(x) = x$ , $\forall$ $x \in X$ is Semi-continuous if and only if $J_1$ is finer than $J_2$ that is $J_2 \subseteq J_1$ .**

**Proof** : The function I is Semi-continuous if and only if for every open set $G \in J_2$, $I^{-1}(G)$ is Semi-open set $\in J_1$ by 2.6. & Since, $I^{-1}(G) = G$ and also G is an open set & every open set is Semi-open set. So, $I^{-1}(G)$ is Semi-open set $\in J_1$.

**Hence, the function I is Semi-continuous if and only if $G \in J_2 \Rightarrow G \in J_1$ i.e., iff $J_2 \subseteq J_1$.**

**2.9 . Proposition : Let $(A , J_A)$ be a subspace of topological space $(X , J)$ then the inclusion function $I : (A , J_A) \longrightarrow (X , J)$ is defined by $I(x) = x$ , $\forall$ $x \in A$ is Semi-continuous.**

**Verification** : Let G be any open subset of X then $I^{-1}(G) = A \cap G$ is Semi-open set in A and A is subspace of set X. **Hence, by 2.6 the inclusion function $I : (A , J_A) \longrightarrow (X , J)$ is defined by $I(x) = x$ , $\forall$ $x \in A$ is Semi-continuous.**

**2.10 . Proposition : Let the function $f : (X , J_1) \longrightarrow (Y , J_2)$ be Semi-continuous & let $A \subseteq X$ then the restriction of function $f$ to A, written as $f_A$ defined by $f_A(x) = f(x)$ $\forall$ $x \in A$ is also Semi-continuous with respect to the R-topology for the set A.**

**Verification** : Let G be any open subset of Y then $f_A^{-1}(G) = A \cap f^{-1}(G)$ and $f^{-1}(G)$ is Semi-open set in X and A is subspace of set X. Hence, $f_A^{-1}(G)$ is Semi-open set in A for the R-topology. **Therefore, the function $f_A$ defined by $f_A(x) = f(x)$ , $\forall$ $x \in A$ is also Semi-continuous with respect to the R-topology for the set A.**

**2.11 . Proposition : Let $(X_1 , J_1)$ and $(X_2 , J_2)$ be two topological spaces then a function $f : (X_1 , J_1) \longrightarrow (X_2 , J_2)$ be Semi-continuous function then the Semi-continuity of $f$ is not destroyed if we replace the topology $J_1$ of X by a finer topology $J_1^*$ or the topology $J_2$ of Y by a weaker topology $J_2^*$.**

**Verification** : Suppose that the function $f : (X_1 , J_1) \longrightarrow (X_2 , J_2)$ be Semi-continuous so for every open set $G \in J_2$, $f^{-1}(G)$ is Semi-open set $\in J_1$ by 2.6.

Now, if $J_1^*$ is finer than $J_1$ , i.e., $J_1 \subseteq J_1^*$ so $f^{-1}(G)$ is Semi-open set $\in J_1^*$.

**Hence, the function $f : (X_1 , J_1^*) \longrightarrow (X_2 , J_2)$ be Semi-continuous so the Semi-continuity of $f$ is not destroyed if we replace the topology $J_1$ of X by a finer topology $J_1^*$.**

Again, for every open set $G \in J_2^*$. Now, if $J_2^*$ is weaker than $J_2$ , i.e., $J_2^* \subseteq J_2$ then $G \in J_2$. Since, the function $f : ( X_1 , J_1) \longrightarrow ( X_2 , J_2)$ be Semi-continuous so $G \in J_2$ then $f^{-1}(G)$ is Semi-open set $\in J_1$.

**Hence, the function $f : (X_1 , J_1^*) \longrightarrow (X_2 , J_2^*)$ be Semi-continuous because $G \in J_2$ $\Rightarrow G \in J_2^*$ then $f^{-1}(G)$ is Semi-open set $\in J_1 \Rightarrow f^{-1}(G)$ is Semi-open set $\in J_1^*$.**

**2.12 . <u>Proposition</u> : Let $(X_1 , J_1)$ and $(X_2 , J_2)$ be two topological spaces then a function $f : ( X_1 , J_1) \longrightarrow ( X_2 , J_2)$ and let $A \subseteq X_1$ & $B \subseteq X_2$ then the following statements are equivalent**

**(1) . $f$ is Semi-continuous**

**(2) . The inverse image of each closed set in $X_2$ is Semi-closed set in $X_1$**

**(3) . $f [ I_N\{C_L (A)\}] \subseteq C_L \{ f(A)]\}$**

**(4) . $I_N [ C_L\{f^{-1}(B)\}] \subseteq f^{-1}\{C_L (B)\}$**

**<u>Verification</u> :**

**First, to prove the equivalence (1) $\Leftrightarrow$ (2)**

**Suppose the function $f$ is Semi-continuous**

Let F be any closed set in $X_2$ so $( X_2 – F )$ is an open set.

Hence, $f^{-1}( X_2 – F )$ is Semi-open set in $X_1$ by definition of Semi-continuous.

But , $f^{-1}( X_2 – F ) = f^{-1}( X_2) – f^{-1}( F ) = X_1 – f^{-1}( F )$ is Semi-open set in $X_1$.

Hence, $f^{-1}( F)$ is Semi-closed set in $X_1$ .

**This proves that (1) $\Rightarrow$ (2)**

**Conversely,**

**Suppose the inverse image of each closed set in $X_2$ is Semi-closed set in $X_1$ .**

If G is any open set in $X_2$ then $(X_2 – G)$ is a closed set in $X_2$ .

Hence, $f^{-1}(X_2 – G)$ is Semi-closed set in $X_1$ .

But, $f^{-1}(X_2 – G) = f^{-1}(X_2) – f^{-1}(G) = X_1 – f^{-1}(G)$ is Semi-closed set in $X_1$.

**Hence, $f^{-1}( G)$ is Semi-open set in $X_1$. So, the function $f$ is Semi-continuous.**

**This proves that (2) $\Rightarrow$ (1).**

**Hence, (1) $\Leftrightarrow$ (2).**

**Next, to prove the equivalence, (1) $\Leftrightarrow$ (3)**

**Suppose the function *f* is Semi-continuous**.

Let $A \subseteq X_1$ and $C_L\{f(A)\}$ is closed set in $X_2$ then $f^{-1}[C_L\{f(A)\}]$ is Semi-closed set in $X_1$.

Now, $f(A) \subseteq C_L\{f(A)\} \Rightarrow A \subseteq f^{-1}[C_L\{f(A)\}]$. ------ (I)

Since, $f^{-1}[C_L\{f(A)\}]$ is Semi-closed set. So, $I_N[C_L\{f^{-1}[C_L\{f(A)\}]] \subseteq f^{-1}[C_L\{f(A)\}]$ .------- (II)

So, by taking the interior and closure of (I),

$I_N\{C_L(A)\} \subseteq I_N[C_L\{f^{-1}[C_L\{f(A)\}]\}]$ . ------(III)

Then from (II) & (III); $I_N\{C_L(A)\} \subseteq I_N[C_L\{f^{-1}[C_L\{f(A)\}]\}] \subseteq f^{-1}[C_L\{f(A)\}]$.

So, $I_N\{C_L(A)\} \subseteq f^{-1}[C_L\{f(A)\}] \Rightarrow$ $\boldsymbol{f[I_N\{C_L(A)\}] \subseteq C_L\{f(A)]\}}$.

**This proves that (1) $\Rightarrow$ (3).**

**Conversely,**

**Suppose $\boldsymbol{f[I_N\{C_L(A)\}] \subseteq C_L\{f(A)]\}}$, for each subset A of $X_1$.** ---- (IV)

Let F be any closed set in $X_2$ and let $A = f^{-1}(F)$ so $f(A) = f[f^{-1}(F)] \subseteq F$. --- (V)

Since, F is a closed set in $X_2$ so $C_L(F) = F$. ---- (VI)

Hence, from (IV) , (V) & (VI) ; $f[I_N\{C_L(A)\}] \subseteq C_L\{f(A)]\} \subseteq C_L(F) = F$.

So, $f[I_N\{C_L(A)\}] \subseteq F \Rightarrow I_N\{C_L(A)\}] \subseteq f^{-1}(F) = A \Rightarrow I_N\{C_L(A)\}] \subseteq A$.

Thus , the set A is Semi-closed set in $X_1$ .

Therefore $A = f^{-1}(F)$ is Semi-closed set in $X_1$ for each closed set F in $X_2$ .

**Hence, F is Semi-continuous.**

**This proves that (3) $\Rightarrow$ (1) .**

**Hence, (1) $\Leftrightarrow$ (3).**

**Finally; to prove the equivalence, (1) $\Leftrightarrow$ (4)**

**Suppose the function *f* is Semi-continuous.**

Let $B \subseteq X_2$ and $C_L(B)$ is closed set in $X_2$ then $f^{-1}[C_L(B)]$ is Semi-closed set in $X_1$.

Since, $B \subseteq C_L(B) \Rightarrow f^{-1}(B) \subseteq f^{-1}\{C_L(B)\}$.---- (VII)

So, by taking the interior and closure of (VII),

$I_N[C_L\{f^{-1}(B)\}] \subseteq I_N[C_L\{f^{-1}\{C_L(B)\}\}]$ . ---- (VIII)

Since, $f^{-1}[C_L(B)]$ is Semi-closed set so $I_N[C_L\{f^{-1}[C_L(B)]\}] \subseteq f^{-1}[C_L(B)]$ . ----- (IX)

Hence, from (VIII) & (IX); $I_N[C_L\{f^{-1}(B)\}] \subseteq I_N[C_L\{f^{-1}\{ C_L (B)\}\}] \subseteq f^{-1}[C_L(B)]$

$\Rightarrow$ **$I_N[C_L\{f^{-1}( B)\}] \subseteq f^{-1}[C_L (B)]$.**

**This proves that (1) $\Rightarrow$ (4).**

**Conversely,**

Suppose, $I_N[C_L\{f^{-1}( B)\}] \subseteq f^{-1}[C_L (B)]$, for each subset B of $X_2$ . ----- (X)

Let F be any closed set in $X_2$ and let B = F in the hypothesis (X),

$I_N[C_L\{f^{-1}( F)\}] \subseteq f^{-1}[C_L (F)] = f^{-1}(F)$. Since, F is a closed set so $C_L (F) = F$.

So, $I_N [C_L\{f^{-1}( F)\}] \subseteq f^{-1}(F)$. Therefore, $f^{-1}(F)$ is Semi-closed set in $X_1$.

**Hence, F is Semi-continuous.**

**This proves that (4) $\Rightarrow$ (1).**

**Hence, consequently (1) $\Leftrightarrow$ (4).**

## REFERENCES : -

**[1]** Levine, N. (1963). **Semi-open sets and semi-continuity in topological spaces**, *Amer. Math. Monthly*, **70**(1), 39-41.

**[2]** Levine, N. (1970). ***Generalized closed sets in topology***, *Rendiconti del Circolo Matematico di Palermo*, **19** (1), 89 - 96.

**[3]** Mashhour, A. S., Hasanein, I. A. & El-Deeb, S. N. (1983). **$\alpha$ −continuous and $\alpha$ −open mappings**, *Acta. Math. Hungar*, **41**, 213-218.

**[4]** Noiri, T. & Ahmad, B. (1982). **A note on semi-open functions**, *Math. Sem. Notes , Kobe Univ. ,* **10**, 437-441.

**[5]** Biswas, N. (1970). **On Characterizations of semi-continuous function**, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **48**, 399-402.

**[6]** Mashhour, A. S., Abd. El-Monsef, M. E. & El-Deeb, S. N., (1982). **On pre-continuous and weak pre-continuous mappings** , *proc. Math. Phys. Soc. Egypt.*, **53**, 47 – 53.

**[7]** Caldas, M., (2000) . **Weak and strong forms of irresolute maps**, *Int. J. Math. And Math. Sci.*, **23**(4) , 253–259.

**[8]** Mahmoud, R. A., & Abd El-Monsef, M. E., (1990). **β-irresolute and β-topological invariant** , *Proc. Pakistan. Acad.Sci.*, **27**, 289–296.

**[9]** Caldas, M., Jafari, S., Noiri, T., & Saraf, R. K., (2005) . **Weak and strong forms of α-irresolute maps** , *Chaos Solitons & Fractals*, **24**(1) , 223–228.

**[10]** Pious Missier, S. & Arul Jesti, J.,( 2012). **A new notion of open-sets in topological spaces**, *Int. J. Math. Arc.*, **3**(11), 3990-3996.

**[11]** Nour T. M., (1995). **Totally semi continuous function**, *Indian J. Pure Appl. Math.,* **7**(26), 675-678.

**[12]** Biswas N.,( 1969) . **Some mappings in topological spaces**, *Bull. Cal. Math. Soc.*, **61**,127-135.

**[13]** Noiri T., (1973). **On semi continuous mappings**, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*,**54**(8) , 210-214.

**[14]** Hamllet N., (1975). **A correction to the paper Semi-open sets and semi-continuity in topological spaces by Norman Levine**, *Proc. Amer. Math. Soc.*, **49**, 458–460.

**[15]** Munkres J. R.,( 1975). **Topology**, *A First Course, Prentice-Hall, Inc.* .

**[16]** Crosseley S. G. (1978). **A note on semi topological properties**, *Proc. Amer. Math. Soc.* **72**, 409 – 412.

**[17]** Kelley J. L., (1955). **General Topology**, *Van Nostrand , New Jersey.*

**[18]** Levine N., (1963). **Some Remarks on closure operator in topological spaces** , *Amer. Math. Monthly,* **70**, 553.

**[19]** Levine N., (1961). **A decomposition of continuity in topological spaces**, *Amer. Math. Monthly*, **68**, 44-46.

**[20]** David R. Wilkins & Hilary, (2001). **Topological spaces**, course 212.

**[21]** Robert, A.& Pious Misser, S., (2012). **A New class of Nearly open sets,** *International Journal of Mathematics Archive*, **3**(7), 1-8.

**[22]** Ibrahim H. Z. Bc., (2013).**Open sets in topological spaces,** *Advances in pure Mathematics*, **3**, 34-40.

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