Hamacher Sum and Hamacher Product of Interval Valued Fuzzy Matrices

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***Abstract*— In this paper, we define two new operations called Hamacher sum and Hamacher product of Interval Valued Fuzzy Matrices(IVFM) and investigate the algebraic properties of Interval Valued Fuzzy Matrices under these operations as well as the properties of Interval Valued Fuzzy Matrices in the case where these new operations are combined with the well-known operations $\wedge$, $\vee$, we have proved some new inequalities connected with Interval Valued Fuzzy Matrices.**

***Keywords*—Fuzzy Matrices, Interval Valued Fuzzy Matrices, Hamacher sum, Hamacher product**

I. INTRODUCTION

We deal with Interval Valued Fuzzy Matrices(IVFM) that is, matrices whose entries are intervals and all the intervals are subintervals of the interval [0,1]. Thomason introduced fuzzy matrices and discussed about the convergence of powers of a fuzzy matrix [10]. Kim and Roush have developed a theory for fuzzy matrices analogous to that for Boolean Matrices [2]. Recently the concept of IVFM a generalization of fuzzy matrix was introduced and developed by Shyamal and Pal [7], by extending the max.min operations on fuzzy algebra Ꞙ =[0,1], for elements a,b ∈Ꞙ, a+b = max{a,b} and a.b = min{a,b}. Among the well-known operations which can be performed on fuzzy matrices are the operations of component wise addition, multiplication, algebraic product, algebraic sum and complement. Much research works are done concerning fuzzy matrices and their applications to medical sciences, engineering, management environment and social sciences. In 1977, Ragab and Emam [6] presented some properties of the min-max composition of fuzzy matrices. Meenakshi [3] studied the theoretical developments of fuzzy matrices. Meenakshi and Kaliraja have represented an IVFM as an interval matrix of its lower and upper limit fuzzy matrices[4]. In [5], Meenakshi and Poongodi have introduced the concept of k –regular interval valued fuzzy matrix and discussed about inverses associated with a k –regular interval valued fuzzy matrix as a generalization of results on regular fuzzy matrix developed in [2]. The operations studied in Shyamal and Pal[8] are extended to intuitionistic fuzzy matrices and studied its algebraic properties by Sriram and Boobalan [9]. Zhang and Zheng [11] introduced bounded sum and bounded product of fuzzy matrices and presented several properties on these operations.

The paper is organized in three sections. We give the basic definitions and operations on fuzzy matrices in section 2 which will be used in this paper. In section 3, we introduce the Hamacher operations on interval valued fuzzy matrices and focusing on its properties. In section 4, the De Morgan's law for the Hamacher operations are established.

II. PRELIMINARIES AND DEFINITIONS

In this section, some basic definitions and results needed are given. Let $(IVFM)_n$ denotes the set of all nxn Interval Valued Fuzzy Matrices.

Definition 2.1

A Fuzzy Matrix(FM) of order mxn is defined as $A=(a_{ij})$, where $a_{ij}\in[0,1]$. Let F_{mn} denote the set of all fuzzy matrices of order mxn.

Definition 2.2

For $A\in F_{mn}$, the transpose is obtained by interchanging its rows and columns and is denoted by A^T.

Definition 2.3

The mxn zero matrix O is the matrix all of whose entries are zero. The nxn identity matrix I is the matrix (a_{ij}) such that $a_{ij} = 1$ if i=j and $a_{ij} = 0$ if $i \neq j$. The mxn universal matrix J is the matrix all of whose entries are 1.

Definition 2.4

Let $A=(a_{ij})$ and $B=(b_{ij})\in F_{mn}$. We write $A \leq B$ if $a_{ij}\leq b_{ij}$ for all I, j and we say that A is dominated by B (or) B dominates A. A and B are said to be comparable if either $A \leq B$ (or) $B \leq A$

Definition 2.5

An Interval Valued Fuzzy Matrix (IVFM) of order mxn is defined as $A=(a_{ij})_{mxn}$, where $a_{ij} = [a_{ijL}, a_{ijU}]$, the ij^{th} element of A is an interval representing the membership value. All the elements of an IVFM are intervals and all the intervals are the subintervals of the interval [0,1].

For $A =(a_{ij}) = ([a_{ijL}, a_{ijU}])$ and $B = (b_{ij}) = ([b_{ijL}, b_{ijU}])$ of order mxn their sum denoted as A+B defined as ,

$$A + B = (a_{ijL}, a_{ijU}) + (b_{ijL}, b_{ijU}) = ([(a_{ijL} + b_{ijL}), (a_{ijU} + b_{ijU})]) \quad \ldots\ldots (2.1)$$

For $A = (a_{ij})_{mxn}$ and $B =(b_{ij})_{nxp}$ their product denoted as AB is defined as,

$$AB = (c_{ij}) = \Sigma\, a_{ik}\, b_{kj} \quad i=1, 2, \ldots, m,\ j=1, 2, \ldots, p \text{ and } k = 1,2, \ldots .n. \ \ldots (2.2)$$

In particular if $a_{ijL} = a_{ijU}$ and $b_{ijL} = b_{ijU}$ then (2.2) reduces to the standard max. min composition of Fuzzy Matrices [1]. $A \leq B$ if and only if $a_{ijL} \leq b_{ijL}$ and $a_{ijU} \leq b_{ijU}$

We define the following operators for any two Interval Valued Fuzzy Matrices $A= (a_{ij})$ and $B = (b_{ij})$ of order mxn,

(i) $A\vee B = (\max(a_{ijL}, b_{ijL}), \max(a_{ijU}, b_{ijU}))$

(ii) $A\wedge B = (\min(a_{ijL}, b_{ijL}), \min(a_{ijU}, b_{ijU}))$

(iii) $A^C = (1- a_{ijL}, 1- a_{ijU})$ (the complement of A)

III. SOME RESULTS ON HAMACHER SUM AND HAMACHER PRODUCT OF INTERVAL VALUED FUZZY MATRICES

Hamacher [1] introduced a generalized *t - norm* and *t - conorm* by defining as

$$T(x,y) = \left(\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}\right) \quad \text{and} \quad T^*(x,y) = \left(\frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy}\right)$$

From these operations, when $\gamma = 0$ they will reduce to algebraic *t - norm* and *t - conorm*,

$$T(x,y) = \left(\frac{xy}{x+y-xy}\right) \quad \text{and} \quad T^*(x,y) = \left(\frac{x+y-2xy}{1-xy}\right)$$

In this section, based on these operations, Hamacher sum and Hamacher product of interval valued fuzzy matrices are defined as

$$A \oplus_H B = \left(\frac{a_{ij} + b_{ij} - 2a_{ij} b_{ij}}{1 - a_{ij} b_{ij}} \right)$$

and $$A \otimes_H B = \left(\frac{a_{ij} b_{ij}}{a_{ij} + b_{ij} - a_{ij} b_{ij}} \right)$$

Lemma 3.1.

For a, b∈[0, 1], $\frac{ab}{a+b-ab} \leq \frac{a+b-2ab}{1-ab}$

Proof: We know that $(a + b)^2 \geq 4ab$ ….(3.1)

$a + b - ab \leq 1 \Rightarrow 1 + 3(a + b - ab) \leq 4$

$\Rightarrow (1 + 3(a + b - ab) \leq 4ab$ ….(3.2)

From (3.1) and (3.2)

$(1 + 3(a + b - ab)ab \leq 4ab \leq (a + b)^2$

$0 \leq (a + b)^2 - (1 + 3(a + b - ab)ab = (a + b)^2 + 3ab(ab - a - b) - ab$

$ab \leq (a + b)^2 + 3a^2b^2 - 3ab(a + b)$

$ab - a^2b^2 \leq (a + b - 2ab)(a + b - ab)$

$\frac{ab}{a+b-ab} \leq \frac{a+b-2ab}{1-ab}$

Hence the Proof.

Definition 3.1.

For any interval valued fuzzy matrices A= $[A_L, A_U]$ and B=$[B_L, B_U]$ of the same size. The Hamacher sum of A and B is defined by

$$A \oplus_H B = [A_L, A_U] \oplus_H [B_L, B_U] = \left(\frac{[a_{ijL}, a_{ijU}] + [b_{ijL}, b_{ijU}] - 2[a_{ijL}, a_{ijU}][b_{ijL}, b_{ijU}]}{1 - [a_{ijL}, a_{ijU}][b_{ijL}, b_{ijU}]} \right)$$

$$= \left(\frac{a_{ij} + b_{ij} - 2a_{ij} b_{ij}}{1 - a_{ij} b_{ij}} \right)$$

The Hamacher product of A and B is defined by

$$A \otimes_H B = [A_L, A_U] \otimes_H [B_L, B_U] = \left(\frac{[a_{ijL}, a_{ijU}][b_{ijL}, b_{ijU}]}{[a_{ijL}, a_{ijU}] + [b_{ijL}, b_{ijU}] - [a_{ijL}, a_{ijU}][b_{ijL}, b_{ijU}]} \right)$$

$$= \left(\frac{a_{ij} b_{ij}}{a_{ij} + b_{ij} - a_{ij} b_{ij}} \right)$$

Next, we have prove some results of properties of Hamacher sum and Hamacher product on IVFMs.

Property 3.1.

Let $A=[A_L, A_U]$ and $B=[B_L, B_U]$ be any two interval valued fuzzy matrices of same size. Then

$$A \otimes_H B \leq A \oplus_H B$$

Proof: By using Lemma 3.1,

$$\left(\frac{a_{ij}b_{ij}}{a_{ij}+b_{ij}-a_{ij}b_{ij}}\right) \leq \left(\frac{a_{ij}+b_{ij}-2a_{ij}b_{ij}}{1-a_{ij}b_{ij}}\right) \text{ for all } i \;\&\; j.$$

Hence, ij^{th} entry of $A \otimes_H B \leq ij^{th}$ entry of $A \oplus_H B$

Therefore, $A \otimes_H B \leq A \oplus_H B$.

Property 3.2.

Let A be any interval valued fuzzy matrix. Then

(i) $A \oplus_H A \geq A$

(ii) $A \otimes_H A \leq A$

Proof:

(i) $A \oplus_H A = \left(\frac{2a_{ij}-2a_{ij}2}{1-a_{ij}2}\right) = \left(\frac{2a_{ij}}{1+a_{ij}}\right) \geq a_{ij}$ for all i, j.

Since $a^2_{ij} - a_{ij} \leq 0 \Rightarrow a^2_{ij} + a_{ij} - 2a_{ij} \leq 0 \Rightarrow a^2_{ij} + a_{ij} \leq 2a_{ij}$.

Hence, ij^{th} entry of $A \oplus_H A \geq ij^{th}$ entry of A.

Therefore, $A \oplus_H A \geq A$.

(ii) $A \otimes_H A = \frac{a_{ij}2}{2a_{ij}-a_{ij}2} = \frac{a_{ij}}{2-a_{ij}} \leq a_{ij}$ for all i, j.

Since $1 \leq 2 - a_{ij}$ (i.e) $a_{ij} \leq \frac{a_{ij}}{2-a_{ij}}$.

Hence, ij^{th} entry of $A \otimes_H A \leq ij^{th}$ entry of A

Therefore, $A \otimes_H A \leq A$.

The following properties are obvious. The operations $\oplus_H$ and $\otimes_H$ are commutative as well as associative. Existence of the identity elements with respect to $\oplus_H$ and $\otimes_H$ are determined in the following Theorems.

Property 3.3.

Let A, B and C be any three interval valued fuzzy matrices of same size. Then

(i) $A \oplus_H B = B \oplus_H A$

(ii) $(A \oplus_H B) \oplus_H C = A \oplus_H (B \oplus_H C)$

(iii) $A \otimes_H B = B \otimes_H A$

(iv) $(A \oplus_H B) \oplus_H C = A \oplus_H (B \oplus_H C)$

Property 3.4.

For a interval valued fuzzy matrix A,

(i) $A \oplus_H O = O \oplus_H A = A$

(ii) $A \otimes_H J = J \otimes_H A = A$

(iii) $A \oplus_H J = J$

(iv) $A \otimes_H O = O$

Thus ($IVFM_{mn}$, $\oplus_H$) and ($IVFM_{mn}$, $\otimes_H$) form commutative monoids. The operators $\oplus_H$ and $\otimes_H$ do not obey the De Morgan's laws over transpose.

Property 3.5.

For any two interval valued fuzzy matrices A and B of same size,

(i) $(A \oplus_H B)^T = B^T \oplus_H A^T$

(ii) $(A \otimes_H B)^T = B^T \otimes_H A^T$

Property 3.6.

For any three interval valued fuzzy matrices A,B and C of same size, if A ≤ B, then $(A \otimes_H C) \leq (B \otimes_H C)$

Proof:

Let $a_{ij} \leq b_{ij}$ for all i, j then,

$$\Rightarrow a_{ij}\, c_{ij}^2 \leq b_{ij}\, c_{ij}^2$$

$$\Rightarrow a_{ij}\, c_{ij}^2 + a_{ij}\, b_{ij}\, c_{ij}\,(1 - c_{ij}) \leq b_{ij}\, c_{ij}^2 + a_{ij}\, b_{ij}\, c_{ij}\,(1 - c_{ij})$$

$$\Rightarrow a_{ij}c_{ij}^2 + a_{ij}b_{ij}c_{ij} - a_{ij}b_{ij}c_{ij}^2 \leq b_{ij}c_{ij}^2 + a_{ij}b_{ij}c_{ij} - a_{ij}b_{ij}c_{ij}^2$$

$$\Rightarrow a_{ij}c_{ij}\,(c_{ij} + b_{ij} - b_{ij}c_{ij}) \leq b_{ij}c_{ij}(c_{ij} + a_{ij} - a_{ij}c_{ij})$$

$$\left(\frac{a_{ij}\, c_{ij}}{a_{ij} + c_{ij} - a_{ij}\, c_{ij}}\right) \leq \left(\frac{b_{ij} c_{ij}}{b_{ij} + c_{ij} - b_{ij} c_{ij}}\right)$$

Therefore, the ij^{th} entry of $A \otimes_H C \leq ij^{th}$ entry of $B \otimes_H C$.

Hence the result.

Property 3.7.

For any two interval valued fuzzy matrices A and B of same size, if A ≤ B, then $A \oplus_H C \leq B \oplus_H C$

Proof: Let $a_{ij} \leq b_{ij}$ for all i,j

$a_{ij}(1 - c_{ij})^2 \leq b_{ij}(1 - c_{ij})^2$

$a_{ij}(1 - 2c_{ij} + c_{ij}^2) \leq b_{ij}(1 - 2c_{ij} + c_{ij}^2)$

$a_{ij} - 2a_{ij}c_{ij} + a_{ij}c_{ij}^2 \leq b_{ij} - 2b_{ij}c_{ij} + b_{ij}c_{ij}^2$

$a_{ij} - 2a_{ij}c_{ij} + a_{ij}c_{ij}^2 + (c_{ij} - a_{ij}b_{ij}c_{ij} + 2a_{ij}b_{ij}c_{ij}^2) \leq b_{ij} - 2b_{ij}c_{ij}b_{ij}c_{ij}^2 + (c_{ij} - a_{ij}b_{ij}c_{ij} + 2a_{ij}b_{ij}c_{ij}^2)$

$a_{ij} + c_{ij} - 2a_{ij}c_{ij} - a_{ij}b_{ij}c_{ij} - b_{ij}c_{ij}^2 + 2a_{ij}b_{ij}c_{ij}^2 \leq b_{ij} + c_{ij} - 2b_{ij}c_{ij} - a_{ij}b_{ij}c_{ij} - a_{ij}c_{ij}^2 + 2a_{ij}b_{ij}c_{ij}^2$

$a_{ij}+c_{ij}-2a_{ij}c_{ij}-b_{ij}c_{ij}(a_{ij}+c_{ij}-2a_{ij}c_{ij}) \leq b_{ij}+c_{ij}-2b_{ij}c_{ij}-a_{ij}c_{ij}(b_{ij}+c_{ij}-2b_{ij}c_{ij})$

$(a_{ij}+c_{ij}-2a_{ij}c_{ij})(1-b_{ij}c_{ij}) \leq (b_{ij}+c_{ij}-2b_{ij}c_{ij})(1-a_{ij}c_{ij})$

$$\left(\frac{a_{ij}+c_{ij}-2a_{ij}c_{ij}}{1-a_{ij}c_{ij}}\right) \leq \left(\frac{b_{ij}+c_{ij}-2b_{ij}c_{ij}}{1-b_{ij}c_{ij}}\right)$$

Therefore, the ij^{th} entry of $A \oplus_H C \leq ij^{th}$ entry of $B \oplus_H C$.

Hence the result.

Property 3.10.

For any two interval valued fuzzy matrices A and B of same size, then

(i) $(A \wedge B) \oplus_H (A \vee B) = (A \oplus_H B)$

(ii) $(A \wedge B) \otimes_H (A \vee B) = (A \otimes_H B)$

Proof:

(i) $(A \wedge B) \oplus_H (A \vee B) = (A \oplus_H B) = (\min(a_{ij}, b_{ij}) \oplus_H (\max(a_{ij}, b_{ij})$

$$= \left(\frac{(\min(a_{ij},b_{ij})+\max(a_{ij},b_{ij})-2\min(a_{ij},b_{ij})\max(a_{ij},b_{ij}))}{(1-\min(a_{ij},b_{ij})\max(a_{ij},b_{ij}))}\right)$$

$$= \left(\frac{(a_{ij}+b_{ij}-2a_{ij}b_{ij}}{1-a_{ij}b_{ij}}\right)$$

$$= (A \oplus_H B)$$

(ii) $(A \wedge B) \otimes_H (A \vee B) = (A \otimes_H B) = (\min(a_{ij}, b_{ij}) \otimes_H (\max(a_{ij}, b_{ij})$

$$= \left(\frac{\min(a_{ij},b_{ij})\max(a_{ij},b_{ij})}{\min(a_{ij},b_{ij})+\max(a_{ij},b_{ij}-\min(a_{ij},b_{ij})\max(a_{ij},b_{ij}))}\right)$$

$$= \left(\frac{a_{ij}b_{ij}}{a_{ij}+b_{ij}-a_{ij}b_{ij}}\right)$$

Hence the result.

IV. Results on complement of interval valued fuzzy matrix

In this section, the complement of a interval valued fuzzy matrix is used to analyse the complementary nature of any system. For example, if A represents the crowdness of the roads at a particular time period then its complement A^c represents the clearness of the roads. Using the following results we can study the complement nature of a system with the help of original interval valued fuzzy matrix. The operator complement obey the De Morgans's law for the operator $\oplus_H$ and $\otimes_H$. This is established in the following property.

Property 4.1. For any two interval valued fuzzy matrices A and B of same size,

(i) $(A \oplus_H B)^c = A^c \otimes_H B^c$

(ii) $(A \otimes_H B)^c = A^c \oplus_H B^c$

(iii) $(A \oplus_H B)^c \leq A^c \oplus_H B^c$

(iv) $(A \otimes_H B)^c \geq A^c \otimes_H B^c$

Proof:

(i) $A^c \otimes_H B^c = (1- a_{ij}) \otimes_H (1- b_{ij})$

$$= \left(\frac{(1-a_{ij})(1-b_{ij})}{(1-a_{ij})+(1-b_{ij})-(1-a_{ij})(1-b_{ij})}\right)$$

$$= \left(\frac{(1-b_{ij}-a_{ij}+a_{ij}b_{ij})}{(1-a_{ij})+(1-b_{ij})-(1+b_{ij}+a_{ij}-a_{ij}b_{ij})}\right)$$

$$= \left(\frac{(1-b_{ij}-a_{ij}+a_{ij}b_{ij})}{2-a_{ij}-b_{ij}-1+b_{ij}+a_{ij}-a_{ij}b_{ij}}\right)$$

$$= \left(\frac{1+a_{ij}b_{ij}-a_{ij}-b_{ij}}{1-a_{ij}b_{ij}}\right)$$

$$= \left(\frac{(1-a_{ij})(1-b_{ij})}{1-a_{ij}b_{ij}}\right)$$

$$= \left(1-\frac{a_{ij}+b_{ij}-2a_{ij}b_{ij}}{1-a_{ij}b_{ij}}\right)$$

$$= (A \oplus_H B)^c$$

(ii) $A^c \oplus_H B^c = (1- a_{ij}) \oplus_H (1- b_{ij})$

$$= \left(\frac{(1-a_{ij})+(1-b_{ij})-2(1-a_{ij})(1-b_{ij})}{1-(1-a_{ij})(1-b_{ij})}\right)$$

$$= \left(\frac{2-a_{ij}-b_{ij}-2(1-b_{ij}-a_{ij}+a_{ij}b_{ij})}{1-(1-b_{ij}-a_{ij}+a_{ij}b_{ij})}\right)$$

$$= \left(\frac{2-a_{ij}-b_{ij}-2+2b_{ij}+2a_{ij}-2a_{ij}b_{ij}}{1-1+b_{ij}+a_{ij}-a_{ij}b_{ij}}\right)$$

$$= \left(\frac{a_{ij}+b_{ij}-2a_{ij}b_{ij}}{a_{ij}+b_{ij}-a_{ij}b_{ij}}\right)$$

$$= \left(1-\frac{a_{ij}b_{ij}}{a_{ij}+b_{ij}-a_{ij}b_{ij}}\right)$$

$$= (A \otimes_H B)^c$$

(iii) From property 1. $A \oplus_H B \geq A \otimes_H B$.

Then $(A \oplus_H B)^c \leq (A \otimes_H B)^c$

$= A^c \oplus_H B^c$.

(iv) $(A \otimes_H B)^c \geq (A \oplus_H B)^c$.

$= A^c \otimes_H B^c$.

Hence the result.

V. Conclusions

The In this article, Hamacher sum and Hamacher product of interval valued fuzzy matrices are defined and some properties are proved. The set of all interval valued fuzzy matrices form a commutative monoids with respect to these operations. Thus the Hamacher sum and Hamacher product are very useful to further works.

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