Nonlinear Cauchy problem for abstract impulsive fractional quasilinear evolution equation with delay

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Abstract

This article derives some sufficient conditions for existence and uniqueness of solution for impulsive fractional integro-differential equations in Banach space using fixed point theorem. Examples added to show application of the result.

1 Introduction

Now a days many of the researchers taking interest in development in the theory of fractional differential equations because of its various applications in science and engineering [1, 2, 3, 4, 5, 6] this is due to its non local property [8] fractional differential equations are considered as an alternative model to nonlinear differential equations [7].Several researchers studied existence and uniqueness of the solutions of fractional order differential equations with classical condition using fixed point theory [5, 8, 9]. Existence results with nonlocal condition studied by N' Guerekata [12], Balachandran and Park [13].

The rapid development toward impulsive differential equations played important role in modeling of many problems [14]. Therefore impulsive differential equations have been great interest to researchers. The existence and uniqueness of impulsive differential equations using fixed point theory studied by many authors [15]. The existence result of impulsive fractional differential equations with classical conditions have been studied by Benchohra and Slimani [16], Mophou [17], Ravichandran and Arjunan [18], Benchohra and Slimani [16]. Balachandran et. al. [19, 20] and Gao et. al. [21]. The systems in which past history of the state is required are modeled into Delay differential equations [22]. Existence and uniqueness of fractional impulsive differential equations with delay was studied by [23].

This article derives some sufficient conditions for impulsive fractional integrodifferential equations of the form

$$\begin{aligned}
{}^{c}D^{\alpha}x(t) &= A(t,x)x(t) + f(t, x(\phi(t)), Tx(t), Sx(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\
\Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\
x(0) &= x_0 - g(x),
\end{aligned}$$

over the interval $[0, T_0]$.

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2 Preliminaries

Some basic definitions and properties of fractional calculus and fractional differential equations used in this article, are as follows:

Definition 2.1. *The Riemann-Liouville fractional integral operator of $\alpha > 0$, of function $f \in L_1(\mathbb{R}_+)$ is defined as*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is gamma function.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is defined as*

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

This derivative has singularity at zero and also requires special initial condition which lacking physical interpretation. To overcome this difficulty, Caputo [24] interchanged the role of operators and defined the fractional derivatives as follows:

Definition 2.3. *The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is defined as*

$${}^{c}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n f(s)}{dt^n} ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Moreover if $0 < \alpha < 1$, then

$${}^{c}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df(s)}{dt} ds.$$

The Riemann-Liouville integral I^{α} and Caputo derivative ${}^{c}D_{0+}^{\alpha}$ satisfies following properties which is mentioned in Kilbas *et. al.* [25] and Samko *et. al.* [26].

Theorem 2.1. *For $\alpha, \beta > 0$ and f has absolutely continuous derivatives up to suitable order then,*
(1)$I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\alpha+\beta} f(t)$
(2)$I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\beta} I_{0+}^{\alpha} f(t)$
(3)$I_{0+}^{\alpha}(f(t)+g(t)) = I_{0+}^{\alpha} f(t) + I_{0+}^{\alpha} g(t)$
(4)$I_{0+}^{\alpha}\ {}^{c}D_{0+}^{\alpha} f(t) = f(t) - f(0), \quad 0 < \alpha < 1$
(5)${}^{c}D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t)$
(6)${}^{c}D_{0+}^{\alpha} f(t) = I_{0+}^{1-\alpha} f'(t), \quad 0 < \alpha < 1$
(7)${}^{c}D_{0+}^{\alpha}\ {}^{c}D_{0+}^{\beta} f(t) \neq {}^{c}D_{0+}^{\alpha+\beta} f(t)$
(8)${}^{c}D_{0+}^{\alpha}\ {}^{c}D_{0+}^{\beta} f(t) \neq {}^{c}D_{0+}^{\beta}\ {}^{c}D_{0+}^{\alpha} f(t)$

Definition 2.4. *Let X be Banach space and $\mathbb{R}_+ = [0,\infty)$. Suppose $f \in L_1(\mathbb{R}_+)$. Let, $C([0,T_0],X)$ be the Banach space of continuous function $x(t)$ with $x(t) \in X$ for $t \in [0,T_0]$ and $||x||_{C([0,T_0],X)} = sup_t||x(t)||$. Let $B(X)$ denote the Banach space of bounded linear operators on X with norm $||A||_{B(X)} = sup\{||A(y)||; y \in X \& ||y|| \leq 1\}$. Also consider,*

$$PC([0,T_0],X) = \big\{x : [0,T_0] \to X; x \in C([t_{k-1},t_k],X),\ and\ x(t_k^-)\ and x(t_k^+)\ exist,\\ k = 1,2,\cdots,p\ with\ x(t_k^-) = x(t_k)\big\},$$

with norm $||x||_{PC} = sup_{t\in[0,T_0]}||x(t)||$. Set $J' = [0,T_0] - \{t_1,t_2,\cdots,t_p\}$.

For convenience, ${}^cD^\alpha_{0+}$ is taken as ${}^cD^\alpha$ and with these definitions and properties, sufficient conditions for existence and uniqueness of solutions are derived as follows:

3 Equation with classical condition

This section presents the study of the existence and uniqueness of the solution of impulsive fractional differential equation with classical condition. Consider the fractional quasilinear impulsive integro-differential equation of the form

$$\begin{aligned} {}^cD^\alpha x(t) &= A(t,x)x(t) + f(t,x(\phi(t)),Tx(t),Sx(t)) \quad t \neq t_k,\ k = 1,2,\cdots,p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1,2,\cdots,p \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

over the interval $[0,T_0]$. Where $A(t,x)$ is bounded quasi linear operator on X and $f : [0,T_0] \times X \times X \times X \to X$, $T,S : X \to X$ are defined by $Tx(t) = \int_0^t h(t,s,x(\psi(s)))ds$ and $Sx(t) = \int_0^{T_0} k(t,s,x(\xi(s)))ds$. Where $h : D_0 \times X \to X,\ D_0 = \{(t,s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \to X,\ D_1 = \{(t,s); 0 \leq t,s \leq T_0\}$ are continuous. The equation (3.1) is equivalent to the integral equation of the form

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}A(s,x(s))x(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}A(s,x(s))x(s)ds \\ +\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}f(s,x(\phi(s)),Tx(s),Sx(s))ds \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,x(\phi(s)),Tx(s),Sx(s))ds + \sum_{0<t_k<t}Ix(t_k^-). \end{cases} \tag{3.2}$$

The following conditions are assumed to show the existence and uniqueness of the solution (3.1).

(H1) $A : [0,T_0] \times X \to X$ is continuous bounded linear operator and there exists a positive constant M, such that $||A(t,x)x - A(t,y)y||_{B(X)} \leq M||x-y||$, for all $x,y \in X$.

(H2) $f : [0,T_0] \times X \times X \times X \to X$ is continuous and there exists positive constants L_1, L_2 and L_3, such that $||f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)|| \leq L_1||x_1-y_1|| + L_2||x_2-y_2|| + L_3||x_3-y_3||$ for all x_1,x_2,x_3,y_1,y_2 and y_3 in X.

(H3) $h : D_0 \times X \to X$ and $k : D_1 \times X \to X$ are continuous and there exists positive constants H and K, such that $||h(t,s,x) - h(t,s,y)|| \leq H||x-y||$ and $||k(t,s,x) - k(t,s,y)|| \leq K||x-y||$ for all x and y in X.

(H4) The functions $I_k : X \to X$ are continuous and there exist positive constants I_k^* for all $k = 1,2,\cdots,p$, such that $||I_kx - I_ky|| \leq I_k^*||x-y||$ for all x and y in X.

Set, $\gamma = \frac{T_0^\alpha}{\Gamma\alpha+1}$ and further assume that,

(H5) $q = \gamma\big[(p+1)[M + L_1 + T_0 H L_2 + T_0 K L_3]\big] + \sum I_k^* \Big\} < 1$

Define $F : PC([0, T_0], X) \to PC([0, T_0], X)$ by

$$Fx(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} A(s, x(s))x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} A(s, x(s))x(s)ds \\ +\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds \\ +\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds + \sum_{0<t_k<t} I_k x(t_k^-). \end{cases} \tag{3.3}$$

Then equation (3.2) has unique solution if F defined by (3.3) has unique fixed point. This means F is well defined bounded operator on $PC([0, T_0], X)$ and F is contraction [28].

Lemma 3.1. *If the operators A, f, T, S and I_k for $k = 1, 2, \cdots, p$ are continuous then F is bounded operator on $PC([0, T_0]X)$.*

Proof. Let a sequence $\{x_n\}$ be converges to x in $PC([0, T_0], X)$. Therefore $||x_n - x|| \to 0$ as $n \to \infty$. Consider,

$$\begin{aligned} ||Fx_n - Fx||_{PC} \leq & \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} ||A(s, x_n(s))x_n(s) - A(s, x(s))x(s)||ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} ||A(s, x_n(s))x_n(s) - A(s, x(s))x(s)||ds \\ & + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} ||f(s, x_n(\phi(s)), Tx_n(s), S_n x(s)) - f(s, x(\phi(s)), Tx(s), Sx(s))||ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} ||f(s, x_n(\phi(s)), Tx_n(s), S_n x(s)) - f(s, x(\phi(s)), Tx(s), Sx(s))||ds \\ & + \sum_{0<t_k<t} ||I_k x_n(t_k^-) - I_k x(t_k^-)|| \end{aligned}$$

Assuming the continuity of A, f, T, S and I_k for $k = 1, 2, \cdots, p$ the right side of above expression tends to zero as $n \to \infty$. Therefore F is continuous on $PC([0, T_0], X)$ and hence F is bounded. □

Now we derive sufficient conditions for existence and uniqueness of the solution of equation (3.1).

Theorem 3.2. *If the hypotheses (H1)-(H5) are satisfied, then the fractional quasi-linear impulsive integro-differential equation (3.1) has unique solution in $PC([0,T_0],X)$ for $0<\alpha\leq 1$.*

Proof. To show equation (3.1) has unique solution it is sufficient to show F defined (3.3) is contraction. Let x and y in $PC([0,T_0],X)$ and consider,

$$\begin{aligned}
||Fx-Fy||_{PC} \leq & \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}||A(s,x(s))x(s)-A(s,y(s))y(s)||ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}||A(s,x(s))x(s)-A(s,y(s))y(s)||ds \\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}||f(s,x(\phi(s)),Tx(s),Sx(s))-f(s,y(\phi(s)),Ty(s),Sy(s))||ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}||f(s,x(\phi(s)),Tx(s),Sx(s))-f(t,y(\phi(s)),Ty(s),Sy(s))||ds \\
&+\sum_{0<t_k<t}||I_kx(t_k^-)-I_ky(t_k^-)||
\end{aligned}$$

Applying hypotheses (H1)-(H4) we obtain,

$$\begin{aligned}
||Fx-Fy||_{PC} \leq & \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}M||x-y||ds+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}M||x-y||ds \\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}\big\{L_1+T_0HL_2+T_0KL_3\big\}||x-y||ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}ds\big\{L_1+T_0HL_2+T_0KL_3\big\}||x-y||ds+\sum_{0<t_k<t}I_k^*||x-y|| \\
\leq & \left\{\frac{T_0^\alpha}{\Gamma(\alpha+1)}\big[(p+1)[M+L_1+T_0HL_2+T_0KL_3]\big]+\sum I_k^*\right\}||x-y|| \\
= & \left\{\gamma\big[(p+1)[M+L_1+T_0HL_2+T_0KL_3]\big]+\sum I_k^*\right\}||x-y||
\end{aligned}$$

Assuming hypotheses (H5) to obtain, $||Fx-Fy||_{PC}\leq q||x-y||$ with $q<1$. Hence by Banach fixed point theorem the equation (3.1) has unique solution. □

4 Equation with nonlocal condition

In this section, classical condition is replaced by a nonlocal condition for existence and uniqueness of solution of the impulsive fractional differential equation.
Consider the fractional quasilinear impulsive integro-differential equation of the form

$$\begin{aligned}
{}^cD^\alpha x(t) &= A(t,x)x(t)+f(t,x(\phi(t)),Tx(t),Sx(t)) \quad t\neq t_k,\ k=1,2,\cdots,p \\
\Delta x(t_k) &= I_k(x(t_k)), \quad t=t_k,\ k=1,2,\cdots,p \qquad (4.1)\\
x(0) &= x_0-g(x)
\end{aligned}$$

over the interval $[0,T_0]$. Where $A(t,x)$is bounded quasi linear operator on X and $f:[0,T_0]\times X\times X\times X\to X$, $T,S:X\to X$ are defined by $Tx(t)=\int_0^t h(t,s,x(\psi(s)))ds$ and $Sx(t)=$

$\int_0^{T_0} k(t,s,x(\xi(s)))ds$. Where $h : D_0 \times X \to X,\ D_0 = \{(t,s); 0 \le s \le t \le T_0\}$ and $k : D_1 \times X \to X,\ D_1 = \{(t,s); 0 \le t, s \le T_0\}$ are continuous and $g : X \to X$ is given function.
The equivalent integral equation of (4.1) is given by

$$x(t) = \begin{cases} x_0 - g(x) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} A(s,x(s))x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} A(s,x(s))ds \\ +\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds \\ +\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds + \sum_{0<t_k<t} Ix(t_k^-). \end{cases} \tag{4.2}$$

The following hypotheses are assumed.

(H6) $g : X \to X$ is continuous and there exist a positive constant g^*, such that $||g(x) - g(y)|| \le g^*||x - y||$ for each x and y in X.

(H7) $q^* = g^* + \gamma\big[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3]\big] + \sum I_k^* < 1$.

Define $G : PC([0,T_0], X) \to PC([0,T_0], X)$ by

$$Gx(t) = \begin{cases} x_0 - g(x) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} A(s,x(s))x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} A(s,x(s))x(s)ds \\ +\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds \\ +\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds + \sum_{0<t_k<t} I_k x(t_k^-). \end{cases} \tag{4.3}$$

Lemma 4.1. *If the operators A, f, T, S and I_k for $k = 1, 2, \cdots, p$ are continuous then G is bounded operator on $PC([0,T_0], X)$.*

Proof. Let a sequence $\{x_n\}$ be converges to x in $PC([0,T_0], X)$. Therefore $||x_n - x|| \to 0$ as $n \to \infty$. Consider,

$$\begin{aligned} ||Gx_n - Gx||_{PC} \le\ & ||g(x_n(s)) - g(x(s))|| + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} ||A(s,x_n(s))x_n(s) - A(s,x(s))x(s)||ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} ||A(s,x_n(s))x_n(s) - A(s,x(s))x(s)||ds \\ & + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} ||f(s, x_n(\phi(s)), Tx_n(s), Sx_n(s)) - f(s, x(\phi(s)), Tx(s), Sx(s))||ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} ||f(s, x_n(\phi(s)), Tx_n(s), Sx_n(s)) - f(s, x(\phi(s)), Tx(s), Sx(s))||ds \\ & + \sum_{0<t_k<t} ||I_k x_n(t_k^-) - I_k x(t_k^-)||. \end{aligned}$$

Assuming the continuity of A, f, T, S, g and I_k for $k = 1, 2, \cdots, p$ the right side of above expression tends to zero as $n \to \infty$. Therefore G is continuous on $PC([0,T_0], X)$ and hence G is bounded. □

Now the sufficient conditions are derived as under for existence and uniqueness of the solution of equation (4.1).

Theorem 4.2. *If the hypotheses (H1)-(H4) and (H6)-(H7) are satisfied, then the fractional quasi-linear impulsive integro-differential equation (4.1) has unique solution in $PC([0,T_0], X)$ for $0 < \alpha \le 1$.*

Proof. To show equation (4.1) has unique solution it is sufficient to show G defined in (4.3) is contraction. Let x and y in $PC([0,T_0],X)$ and consider,

$$\begin{aligned}||Gx-Gy||_{PC} \leq ||g(x)-g(y)|| &+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}||A(s,x(s))x(s)-A(s,y(s))y(s)||ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}||A(s,x(s))x(s)-A(s,y(s))y(s)||ds\\ &+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}||f(s,x(\phi(s)),Tx(s),Sx(s))-f(s,y(\phi(s)),Ty(s),Sy(s))||ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}||f(s,x(\phi(s)),Tx(s),Sx(s))-f(t,y(\phi(s)),Ty(s),Sy(s))||ds\\ &+\sum_{0<t_k<t}||I_kx(t_k^-)-I_ky(t_k^-)||\end{aligned}$$

Applying hypotheses (H1)-(H4) and (H6) the result is,

$$\begin{aligned}||Gx-Gy||_{PC} \leq g^*||x-y|| &+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}M||x-y||ds+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}M||x-y||ds\\ &+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\alpha-1}\{L_1+T_0HL_2+T_0KL_3\}||x-y||ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}ds\{L_1+T_0HL_2+T_0KL_3\}||x-y||ds+\sum_{0<t_k<t}I_k^*||x-y||\\ &\leq\left\{g^*+\frac{T_0^{\alpha}}{\Gamma(\alpha+1)}\big[(p+1)[M+L_1+T_0HL_2+T_0KL_3]\big]+\sum I_k^*\right\}||x-y||\\ &=\left\{g^*+\gamma\big[(p+1)[M+L_1+T_0HL_2+T_0KL_3]\big]+\sum I_k^*\right\}||x-y||\end{aligned}$$

Assuming hypotheses (H7) to obtain, $||Gx-Gy||_{PC}\leq q^*||x-y||$ with $q^*<1$. Hence by Banach fixed point theorem the equation (4.1) has unique solution. □

Example 4.2.1. *Consider the following fractional integro-differential equation with the impulsive condition,*

$$\begin{aligned}{}^cD^{\alpha}x(t) &= \frac{1}{9}sinx(t)x(t)+\frac{1}{(t+3)^4}\frac{|x(sint)|}{1+|x(sint)|}+\frac{1}{9}\int_0^t se^{\frac{-x(coss)}{4}}+\frac{1}{9}\int_0^1(t-s)x^2ds\\ \Delta x(1/2) &= \frac{|x(1/2^-)|}{18+|x(1/2^-)|}\\ x(0) &= x_0-\frac{x}{18}\end{aligned}\tag{4.4}$$

where $\alpha=1/2$ *over the interval* $[0,1]$.

Since, $A(t,x)=\frac{1}{9}sinxI$ therefore $||A(t,x)x-A(t,y)y||\leq\frac{1}{18}||sinxIx-sinyIy||\leq\frac{1}{18}||x-y||$, $||Tx-Ty||\leq\frac{1}{9}\int_0^t s||e^{\frac{-x(sins)}{4}}-e^{\frac{-y(sins)}{4}}||\leq\frac{1}{8}||x-y||$, and $||Sx-Sy||\leq\frac{1}{9}\int_0^1|(t-s)||x^2-y^2|||ds\leq\frac{1}{18}||x-y||$ therefore $q^*=g^*+\gamma\big[(p+1)[M+L_1+T_0HL_2+T_0KL_3]\big]+\sum I_k^*<1$. Therefore by existence theorem the given system has unique solution in the interval $[0,1]$.

5 Remark

1. This method suggest not only the existence and uniqueness about the solution but it also suggest method to find approximate solution of impulsive fractional differential equations (3.1) and (4.1).

2. This condition is not necessary condition this means equations (3.1) and (4.1) may have solution if one of the (H1) to (H7) not satisfied.

6 Conclusion

The system taken by Balachandran et. al. [23] is a special case of the system taken in this paper because of inclusion of the nonlinear Fredholm operator in the system which is more relevant in many physical situations.

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