# ON THE HOMOGENEOUS BI-QUADRATIC EQUATION WITH FIVE UNKNOWNS

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## ABSTRACT

The Bi-quadratic equation with five unknowns given by $4x^2 - 4y^2 = 34(4z^2 - 4w^2)T^2$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

**Keywords :** Bi-Quadratic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers.

## INTRODUCTION

Bi-quadratic Diophantine Equations(homogeneous and non-homogeneous) have aroused the interest of numerous mathematicians since ambiguity as can be seen from [1 – 7]. In the context one may refer [8 – 16] for varieties of problems on the Diophantine equations with two, three and four variables. This communication concerns with the problems of determining non-zero integral solutions of yet another bi-quadratic equation in five unknowns represented by $4x^2 - 4y^2 = 34(4z^2 - 4w^2)T^2$. A few interesting relations between the solutions and special polygonal numbers are presented.

## NOTATIONS

- $t_{m,n}$ = Polygonal number of rank n with size m
- $P_m^n$ = pyramidal number of rank n with size m
- $Pr_n$ = pronic number of rank n
- $SO_n$ = Stella octangular number of rank n
- $j_n$ = jacobthal lucas number of rank n
- $J_n$ = Jacobthal number of rank n

- $Gno_n$ = Gnomic number of rank n
- $Cp_n^6$ = Centered pyramidal number of rank n with size m
- $Cp_n^{14}$ = Centered tetra decagonal pyramidal number of rank n
- $Ky_n$ = Kynea number of rank n.

**METHOD OF ANALYSIS**

The Diophantine equation representing the Bi – Quadratic equation with five unknowns reader consideration is

$$4X^2 - 4Y^2 = 34(4Z^2 - 4W^2)T^2 \quad \text{---------------} (1)$$

on substituting the linear transformation

$$x = 2u + 2v,\ y = 2u - 2v,\ z = 4u + 4v,\ w = 4u - 4v \quad \text{---------}(2)$$

In (1) leads to

$$u^2 + v^2 = 17T^2 \quad \text{----------------} (3)$$

**PATTERN – I**

In equation (3) can be written as,

$$u^2 + v^2 = 17T^2$$

$$u^2 - T^2 = 16T^2 - v^2$$

$$(u + T)(u - T) = (4T + v)(4T - v)$$

$$\frac{u+T}{4T+v} = \frac{4T-v}{u-T} = \frac{a}{b} \quad \text{----------------} (4)$$

This is equivalent to the system of derivable equations

$$-ua + Ta - vb + 4Tb = 0$$

$$-ua - vb + T(a + 4b) = 0$$

Applying the method of cross multiplication, we have

$$\left.\begin{aligned} u &= -a^2 + b^2 - 8ab \\ v &= 4a^2 - 4b^2 - 2ab \\ T &= -(a^2 + b^2) \end{aligned}\right\} \quad \text{---------------}(5)$$

Substituting the values of u and v, we get the non - zero integer solutions are,

$$x = 6a^2 - 6b^2 - 20ab$$

$$y = -10a^2 + 10b^2 - 12ab$$

$$z = 12a^2 - 12b^2 - 40ab$$

$$w = 20a^2 + 20b^2 - 40ab$$

$$T = -(a^2 + b^2)$$

**PROPERTIES :**

1. $z(a(a+1),a+2)-12T(a(a+1),a+2)-288FN_a^4-3T_{4,4a}$
$$-48CP_a^6+240P_a^3=0$$
2. $T(3a^2-1,a^2)+10T_{4,a^2}-6T_{4,a}+1=0$
3. $x(2a^2-1,a)-288FN_a^4+T_{14,a}+20So_a\equiv 6\ (mod\ 5)$
4. $y(a^2,a+1)+10T_{4,a^2}-10(obl)_a+24PP_a\equiv 10\ (mod\ 10)$
5. $x(a(a+1),a)-6T(a(a+1),a)-144FN_a^4-6T_{4,2a}$
$$-24CP_a^6+40P_a^5=0$$
6. $T(a^2,a+1)-17T_{4,a^2}-T_{4,a}-136PP_a\equiv 1\ (mod\ 2)$
7. $w(a,4a-3)+12T(a,4a-3)+32T_{4,2a}+40T_{10,a}\equiv 72\ (mod\ 192)$

**PATTERN – II**

Assume $T=T(a,b)=a^2+b^2$ --------------- (6)

Write $17=(1+4i)(1-4i)$ --------------- (7)

Using (6) and (7) in (3) and employing factorization it is expressed as which is equivalent to the system of equations

$$(u+iv)=(1+4i)(a+ib)^2$$
$$(u-iv)=(1-4i)(a-ib)^2 \quad \text{-------------- (8)}$$

Equating the real and imaginary parts, we have

$$u=a^2-b^2-8ab$$
$$v=4a^2-4b^2+2ab \quad \text{-------------- (9)}$$

Substituting (9) in (2), we get integer solutions

$$x=10a^2-10b^2-12ab$$
$$y=-6a^2+6b^2-20ab$$
$$z=12a^2-12b^2-40ab$$
$$w=-20a^2+20b^2-24ab$$
$$T=-(a^2+b^2)$$

**PROPERTIES :**

1. $2x(a(a+1),2a+1)+w(a(a+1),2a+1)-288P_a^4=0$
2. $2y(a,5a-3)+z(a,5a-3)+160T_{7,a}=0$
3. $2x(a(a+1),5a-2)+w(a(a+1),5a-2)+288HP_a=0$
4. $z(a,2a^2-1)-12Pr_a-576(4FD_a)+40So_a\equiv 12(mod\ 12)$
5. $y(1,b)-6T(1,b)+24T_{3,b}+20Pr_b-20T_{4,b}=0$
6. $x(a,2a-1)+10T(a,2a-1)-5T_{4,2a}+12T_{6,a}=0$
7. $2y(a^2,a+1)+z(a^2,a+1)+160PP_a=0$
8. $2x(a,2a^2+1)+w(a,2a^2+1)+144\,OH_a=0$
9. $y(a,2a^2+1)-6T_{4,2a^2}-9So_a+60\,OH_a\equiv 6(mod\ 9)$

10. $2y(a,5a-1)+80T_{12,a}=0$

**PATTERN – III**

Write 17 as

$17=(4+i)(4-i)$ -------------- (10)

$17=(4+i)(a+ib)^2$

$=(4a^2-4b^2-2ab)+i(a^2-b^2+8ab)$

Following a similar procedure as in pattern II the solution for (3) as follow

$u=4a^2-4b^2-2ab$

$v=a^2-b^2+8ab$ --------------- (11)

In view of (2) and (11) the solution of (1) are obtained as

$x=10a^2-10b^2+12ab$

$y=6a^2-6b^2-20ab$

$z=20a^2-20b^2+24ab$

$w=12a^2-12b^2-40ab$

$T=-(a^2+b^2)$

**PROPERTIES :**

- $3x(a(a+1),b+2)-5y(a(a+1),b+2)-816Tet_a=0$
- $x(a,a+1)+10T(a,a+1)-5T_{4,2a}-24T_{3,a}=0$
- $y(2a+1,2a)+80T_{4,a}\equiv 6(mod\ 16)$
- $10T(b^2,b^2-1)+x(b^2,b^2-1)-w(b^2,b^2-1)-240(4DF_b)+T_{4,2a}$ $-624FN_b^4-12=0$
- $3z(a(a+1),5a-2)-5w(a(a+1),5a-2)-1632P_a^7=0$
- $6x(a,2a^2-1)-10y(a,2a^2-1)-272So_a=0$
- $w(a(a+1),2a+1)-12T(a(a+1),2a+1)+6T_{4,2a}+48CP_a^6$ $+240P_a^4+24=0$
- $12x(7b^3-4b,1)-20y(7b^3-4b,1)-1632CP_b^{14}=0$
- $z\big(a(a+1),(a+2)(a+3)\big)-5w\big(a(a+1),(a+2)(a+3)\big)$ $-544PT_a=0$
- $12x\big(a(a+1),(a+2)\big)-20\big(a(a+1),(a+2)\big)-6528TH_a=0$

**PATTERN - IV**

From equation (3)

$17T^2 - v^2 = u^2 * 1$ --------------- (12)

Assume $u = 17a^2 - b^2$ --------------- (13)

Write $1 = (\sqrt{17} - 4)(\sqrt{17} + 4)$ --------------- (14)

Using (13) and (14) in (12) and employing the method of factorization,

$(\sqrt{17} + v) = (\sqrt{17} + 4)(\sqrt{17} + b)^2$

$(\sqrt{17} + v) = \sqrt{17}(17a^2 + b^2 + 8ab) + (68a^2 + 4b^2 + 34ab)$

Equating the rational and irrational parts, we have

$v = 68a^2 + 4b^2 + 34ab$

$T = 17a^2 + b^2 + 8ab$ --------------- (15)

In view of (2) and (15), the solution of (1) are obtained as,

$x = 170a^2 + 6b^2 + 68ab$

$y = -102a^2 - 10b^2 - 68ab$

$z = 340a^2 + 12b^2 + 136ab$

$w = -204a^2 - 20b^2 - 136ab$

$T = 17a^2 + b^2 + 8ab$

**PROPERTIES :**

1. $y(b, 4) + 102Pr_b \equiv 160(mod\ 170)$
2. $x(a, 2a - 1) - 388T_{3,a} - 68T_{6,a} \equiv 6(mod\ 218)$
3. $y(a, 2a^2 - 1) + 6T(a, 2a^2 - 1) + 16T_{4,2a^2} + 16T_{4,a} + 4So_a + 4 = 0$
4. $T(a, a + 1) - 18T_{4,a} - 16T_{3,a} \equiv 1(mod\ 2)$
5. $z(a(a + 1), 4a - 1) - 4080FN_a^4 - 132T_{4,2a} - 680CP_a^6$ $-816P_a^6 \equiv 12(mod\ 96)$
6. $T(a^2, a + 1) - 17T_{4,a^2} - T_{4,a} - 136PP_a \equiv 1(mod\ 2)$
7. $w(a, 4a - 3) + 12T(a, 4a - 3) + 32T_{4,2a} + 40T_{10,a} \equiv 72(mod\ 192)$

**PATTERN - V**

Introduction to the linear transformation

$v = X + 17R, \quad T = X + R, \quad u = 4U$ -------------- (16)

Equation (3) can be written as

$X^2 = 17R^2 - V^2$

$X = r^2 + 17s^2, \ R = 2rs, \quad U = r^2 - 17s^2$

Thus we have

$x = 10r^2 - 102s^2 + 68rs$

$y = 6r^2 - 170s^2 - 68rs$

$z = 20r^2 - 204s^2 + 136rs$

$w = 12r^2 - 340s^2 - 136rs$

$T = r^2 + 17s^2 + 2rs$

**PROPERTIES :**

1. $x(r^2, r+1) + 120FN_r^4 + 112\, obl_r - 136P_r^5 \equiv 102\ (mod\ 92)$
2. $z(r, 3r-1) + 1816T_{4,r} - 272T_{5,r} - 204Gno_r \equiv 0(mod\ 816)$
3. $T(r(r+1), r+2) - 124DF_r - 38PP_r + 17CP_r^6 - 4P_r^3 \equiv 68(mod\ 68)$
4. $y(r, (r+1)(r+2)) - 6T\big(r, (r+1)(r+2)\big) + 272T_{4,r} + 480Tet_r = 0$
5. $w\big(2r+1, r(r+1)\big) + 20T\big(2r+1, r(r+1)\big) - 128\, obl_r + 96P_r^4 = 32$
6. $z(r, (r+1)(5r-2) + 12T(r, (r+1)(5r-2) - 2T_{4,4r} - 960HP_r = 0$
7. $x(2,0) + 6T(2,0) - CP_r^6 = 0$
8. $x(r, 3r-2) - 16T_{4,r} - 80T_{8,r} = 0$
9. $z(r^2, 2r) + 120T(r^2, 2r) - 32\, T_{4,r^2} - 320CP_r^6 = 0$

**PATTERN – VI**

We can write 17 as

$$17 = \frac{(20+i5)(20-i5)}{25} \qquad \text{--------------- (17)}$$

The corresponding integer solutions are given by

$u = 4a^2 - 4b^2 - 2ab$

$v = a^2 - b^2 - 8ab$ ---------------- (18)

Thus the corresponding solutions of (1) are found to be

$x = 10a^2 - 10b^2 + 12ab$

$y = 6a^2 - 6b^2 - 20ab$

$z = 20a^2 - 20b^2 + 24ab$

$w = 12a^2 - 12b^2 - 40ab$

$T = -(a^2 + b^2)$

**PROPERTIES :**

1. $x\big(a(a+1), (a+2)\big) - 10T_{4,a^2} - 20CP_a^6 - 72TH_a \equiv 40(mod\ 40)$
2. $6x(a(a+1), 5a-2) - 10y(a(a+1), 5a-2) - 552P_a^7 = 0$
3. $6z(a, 13a-11) - 5w(a, 13a-11) - 544T_{15,a} = 0$
4. $y(a,2) - S_a + 3J_4 + j_3 \equiv 3(mod\ 34)$
5. $x(a,1) - 10Pr_a - Gno_a + 3J_3 = 0$
6. $z(a^2, a) - 204DF_a - 24CP_a^6 = 0$
7. $x(a,5) - 6T_{4,a} \equiv 150\ (mod\ 100)$

**CONCLUTION :**

In this paper we have presented different patterns of non – zero integer solution of the Bi-Quadratic equation with five unknowns. One may search for other patterns of solutions and their corresponding properties.

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