A Study on Bipolar Fuzzy Vietories Topological Spaces

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Abstract

The main objective of this paper is to introduce new spaces called bipolar fuzzy topological spaces and bipolar fuzzy vietories topological spaces. We define a bipolar fuzzy T_0 and T_1 space and examine some its characterizations of these spaces. Finally, we define a bipolar fuzzy separable.

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I. INTRODUCTION

American cyberneticst Lofti A. Zadeh [7] introduced the fuzzy set in 1965. The notion of a fuzzy topology is established by Chang [1] in 1968. Lee [3] found the idea of bipolar fuzzy sets which is an extension of traditional, fuzzy sets whose membership degree range is extended from [0, 1] to [-1, 1] and also in 1994, Zhang [8] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets . A fuzzy vietories topology is developed by K. Hur, J.R. Moon and J.H. Ryou [2] in 2000. In this article, a new class of space called bipolar fuzzy topological space, bipolar fuzzy vietories topological space and bipolar fuzzy T_0 and T_1 spaces are introduced and studied some of its properties. we investigate the equivalent relation between bipolar fuzzy exponential and bipolar fuzzy vietories topology. Finally, we define a bipolar fuzzy separable space and discussed its properties.

II. PRELIMINARIES

Definition 2.1[6]

Given a point $(a,b)\in X\times X$, the bipolar valued fuzzy set $f=(X,f_n,f_p)$ defined by $f(x,y)=\begin{cases}(\alpha,\beta)\ if\ (x,y)=(a,b)\\ (0,0)\ if\ (x,y)\neq(a,b)\end{cases}$ where $(\alpha,\beta)\in[-1,0)\times(0,1]$ is called as bipolar valued fuzzy point with support (a, b) and value (α,β) and is denoted by $\langle(a,b);(\alpha,\beta)\rangle$. Also denote by a_α and (b,β) we mean an *N*-point and a fuzzy point respectively.

Definition 2.2[8]

Let G be a non-empty set and $c\in G$ be fixed. If $\gamma\in(0,1]$ and $\delta\in[-1,0)$ are two real numbers then $c(\gamma,\delta)=\langle x,c_\gamma,c_\delta\rangle$ is called a bipolar fuzzy point in G, where γ is the positive degree of membership and δ is the negative degree of membership function of $c(\gamma,\delta)$ and $c\in G$ is a support of $c(\gamma,\delta)$. Let $c(\gamma,\delta)$ be a bipolar fuzzy point in G and let $A=\langle x,\mu_A{}^P,\mu_A{}^N\rangle$ be a bipolar fuzzy set in G. Then bipolar fuzzy point $c(\gamma,\delta)$ is said to belong to bipolar fuzzy set a, $c(\gamma,\delta)\in A$ if $\mu_A{}^P(c)\geq\gamma$, $\mu_A{}^N(c)\leq\delta$. We say that bipolar fuzzy point is said to be quasi coincident bipolar fuzzy set $c(\gamma,\delta)qA$ if $\mu_A{}^P(c)+\gamma>1$ and $\mu_A{}^N(c)+\delta<-1$.

Definition 2.3[3]

Let S be a Universe of discourse. A bipolar –valued fuzzy set A in S is an object with the form $A=\{\langle x,\mu^P(x),\mu^N(x)\rangle / x\in S\}$, where μ^P is a positive map from S into [0, 1], and μ^N is a negative map from S into [-1, 0]. For the sake of simplicity, we shall use the symbol $A=(\mu_A{}^P,\mu_A{}^N)$ expressing the bipolar-valued fuzzy set $A=\{\langle x,\mu^P(x),\mu^N(x)\rangle / x\in S\}$, $\mu_A{}^P$ which expresses the satisfaction degree of an element $x\in S$ about some property is often called a positive membership degree, $\mu_A{}^N$ which expresses the satisfaction degree of an element $x\in S$ about some implicit counter-property is often called a negative membership degree.

Definition 2.4[5]

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy dense if there exists no fuzzy closed set μ in (X, T) such that λ < μ < 1 That is cl(λ) = 1.

III. BIPOLAR VALUED FUZZY TOPOLOGICAL SPACE

Notation

(i) $J_{[0,1]}=I_+{}^X$.

(ii) $J_{[-1,0]}=I_-{}^X$.

Remark 3.1

Let $I=[0,1]$ and $I_0=(0,1]$. Let $I_+{}^X$ be the collections of all mappings from $X\to[0,1]$, $I_-{}^X$ be the collections of all mappings from $X\to[-1,0]$. Let $I_{0+}{}^X$ be the collections of all mappings from $X\to(0,1]$, $I_{0-}{}^X$ be the collections of all mappings from $X\to[-1,0)$. Let $\mathcal{B}^X = I_+{}^X\times I_-{}^X$ be the collection of all bipolar valued fuzzy set $A=\{\langle x,\mu^P(x),\mu^N(x)\rangle / x\in X\}$ in X can be identified by an ordered pair $\langle\mu^P,\ \mu^N\rangle$ in $\mathcal{B}^X$ or by an element in $\mathcal{B}^X$. Let $\mathcal{B}_0{}^X = I_{0+}{}^X\times I_{0-}{}^X$ be the collection of all bipolar valued fuzzy set $A=\{\langle x,\mu^P(x),\mu^N(x)\rangle / x\in X\}$ in X can be identified by an ordered pair $\langle\mu^P,\ \mu^N\rangle$ in $\mathcal{B}_0{}^X$ or by an element in $\mathcal{B}_0{}^X$.

Definition 3.1

$$0_{+_\sim}=\{\langle x,0,0\rangle : x\in X\},\ 1_{+_\sim}=\{\langle x,1,-1\rangle : x\in X\}$$

Definition 3.2

A bipolar valued fuzzy topology on a non-empty set X is a collection $\tau_{\mathcal{BF}}\subseteq\mathcal{B}^X$ of Bipolar valued fuzzy subsets (in short BPVSs) in X satisfying the following conditions

i) $0_{+_\sim}, 1_{+_\sim}\in\tau_{\mathcal{BF}}$.

ii) $A_1\cap A_2\in\tau_{\mathcal{BF}}$ For any A_1 and $A_2\in\tau_{\mathcal{BF}}$.

iii) $\bigcup G_i\in\tau_{\mathcal{BF}}$ for any arbitrary family $\{G_i : i\in J\}\subseteq\tau_{\mathcal{BF}}$.

The pair $(X,\tau_{\mathcal{BF}})$ is called bipolar valued fuzzy topological space (in short BPVFTS). Any bipolar valued fuzzy set in $\tau_{\mathcal{BF}}$ is known as bipolar valued fuzzy open set in X and its complement is bipolar valued fuzzy closed set.

Example 3.1

Let X = {a, b} and Let $A = \left\langle x, \frac{a}{\langle 0.5, -0.3\rangle}, \frac{b}{\langle 0.1, -0.7\rangle} \right\rangle$ $B = \left\langle x, \frac{a}{\langle 0.7, -0.2\rangle}, \frac{b}{\langle 0.3, -0.5\rangle} \right\rangle$

$A \bigcup B = \left\langle x, \frac{a}{\langle 0.7, -0.3\rangle}, \frac{b}{\langle 0.3, -0.7\rangle} \right\rangle$, $A \bigcap B = \left\langle x, \frac{a}{\langle 0.5, -0.2\rangle}, \frac{b}{\langle 0.1, -0.5\rangle} \right\rangle$ where $x \in X$. Then the family $\tau_{\mathcal{BF}} = \{0_{+_\sim}, 1_{+_\sim}, A, B, A \bigcup B, A \bigcap B\}$ forms a bipolar valued fuzzy topology. Then the pair $(X, \tau_{\mathcal{BF}})$ is called a BPVFTS.

Result 3.1

Let A and $B \in \mathcal{B}^X$ and $c(\gamma, \delta)$ be a bipolar fuzzy point. Then the following are equivalent for any two bipolar valued fuzzy sets

i) $A \subset B$.
ii) $c(\gamma, \delta) \subset B$ for all $c(\gamma, \delta) \subset A$.
iii) $c(\gamma, \delta) \in B$ If $(\mu_A^P(c) \geq \gamma, \mu_A^N(c) \leq \delta)$ for all $c \in G$, where G is a non empty set and $c(\gamma, \delta) \in A, (\mu_A^P(c) \geq \gamma, \mu_A^N(c) \leq \delta)$.

Proof: It is easy by using definition 2.2

Definition 3.3

Let A be a bipolar fuzzy set in a BPVFTS $(X, \tau_{\mathcal{BF}})$, then the bipolar fuzzy closure of A is and the bipolar fuzzy interior of A is said to be defined as

$$BFCl(A) = \overline{A} = \bigcap\{B : A \subseteq B, B^c \in \tau_{\mathcal{BF}}'\}, \quad BFInt(A) = \overset{\circ}{A} = \bigcup\{B : B \subseteq A, B \in \tau_{\mathcal{BF}}\}.$$

Result 3.2

Let $(X, \tau_{\mathcal{BF}})$ be aBPVFTS and let $A \in \mathcal{B}^X$. Then $\overset{\circ}{A} = \left((\overline{A^c})\right)^c = (BFCl(A^c))^c$ and $BFCl(A) = \overline{A} = ((\overset{\circ}{A})^c)^c = ((BFIntA)^c)^c$.

Proof:

Let $BFInt(A) = \overset{\circ}{A} = \bigcup\{B_j : B_j \subset A, B_j \in \tau_{\mathcal{BF}}\}$ and since, we know that

$(BFInt(A))^C = \overset{\circ}{A}^C = (\bigcup\{B_j : B_j \subset A, B_j \in \tau_{\mathcal{BF}}\})^C = \bigcap\{E_j : E_j \supset A^C, E_j \in \tau_{\mathcal{BF}}'\}$ is the family of all the closed sets containing $A^C = \{\langle x, \mu_{A^c}{}^P(x), \mu_{A^c}{}^N(x)\rangle / x \in X\}$ where $\mu_{A^c}{}^P(x) = 1 - \mu_A{}^P(x), \mu_{A^c}{}^N(x) = -1 - \mu_A{}^N(x)$. Thus, $((BFInt(A))^c)^c = BFCl(A)$. Hence, $BFCl(A^c) = \overline{A^c} = \{\bigcap\left(\overset{\circ}{A}\right)^C\}$. By DeMorgan's law, we have $BFCl(A^c) = (\bigcap(A^c))^c = \bigcup\{(A^c)^c\} = \bigcup \overset{\circ}{A} = \overset{\circ}{A}$.

Definition 3.4

A bipolar fuzzy set A in $(X, \tau_{\mathcal{BF}})$ is called a bipolar fuzzy neighborhood of a bipolar fuzzy point $c(\gamma, \delta)$ if and only if $\exists B \in \tau_{\mathcal{BF}} \ni c(\gamma, \delta) \in B \subseteq A.$ A bipolar fuzzy neighborhood A is said to be bipolar fuzzy open if and only if A is bipolar fuzzy open.

Definition 3.5

Let $\mathcal{N}^{c(\gamma,\delta)}$ be the collection of all the neighborhoods of $c(\gamma, \delta),$ for each bipolar fuzzy set belonging to $\mathcal{N}^{c(\gamma,\delta)}$ be said to be the system of bipolar fuzzy neighborhoods of $c(\gamma, \delta)$.

(i.e.) $\mathcal{N}^{c(\gamma,\delta)} = \{A \in BPFS(X) : B \in \tau_{\mathcal{BF}}, c(\gamma, \delta) \in B \subseteq A\}.$

Definition 3.6

A Bipolar fuzzy set A in $(X, \tau_{\mathcal{BF}})$ is said to be Q-bipolar fuzzy neighborhood of $c(\gamma, \delta)$ if and only if $\exists B \in \tau_{\mathcal{BF}}, c(\gamma, \delta) q B \subseteq A$. The family consisting of all the Q-bipolar fuzzy neighborhoods of $c(\gamma, \delta)$ is called the system of Q-bipolar fuzzy neighborhoods of bipolar fuzzy point $c(\gamma, \delta)$. Let $\mathcal{N}_{Q}^{c(\gamma,\delta)}$ denotes the family of all open Q-bipolar fuzzy neighborhoods of a bipolar fuzzy point $c(\gamma, \delta)$ in X (i.e.) $\mathcal{N}_{Q}^{c(\gamma,\delta)} = \{A \in BPVFS(X) : B \in \tau_{\mathcal{BF}}, c(\gamma, \delta) q B \subseteq A\}$. A bipolar fuzzy point $c(\gamma, \delta) \in BFCl(A)$ if and only if every Q-bipolar fuzzy neighborhood of $c(\gamma, \delta)$ is quasi coincident with A. A bipolar valued fuzzy point $c(\gamma, \delta) \in BFInt(A)$ if and only if it has a bipolar fuzzy neighborhood B contained in A. i.e. A bipolar fuzzy neighborhood A is said to be open if and only if A is bipolar fuzzy open.

Definition 3.7

Let $(X, \tau_{\mathcal{BF}})$ be a BPVFTS. Then a sub collection $\mathcal{B}$ of $\tau_{\mathcal{BF}}$ is called a bipolar fuzzy base for $\tau_{\mathcal{BF}}$ if for every $A \in \tau_{\mathcal{BF}}, \exists \{A_j : \forall\, j \in J\} \subset \mathcal{B} \ni A = \bigcup_{j \in J} (A_j) = \left\langle x, \vee_{j \in J} \mu_{A_j}^{P}, \wedge_{j \in J} \mu_{A_j}^{N} \right\rangle$ is in $\tau_{\mathcal{BF}}$.

Definition 3.8

Let $(X, \tau_{\mathcal{BF}})$ be a BPVFTS. A sub collection S of $\tau_{\mathcal{BF}}$ is called a bipolar fuzzy subbase for $\tau_{\mathcal{BF}}$ if the family of finite intersection of bipolar fuzzy members of S forms a bipolar fuzzy base for $\tau_{\mathcal{BF}}$.

Definition 3.9

A BPVFTS $(X, \tau_{\mathcal{BF}})$ is said to satisfy the first axiom of countability or is said to be a bipolar fuzzy C_I space if only if $\tau_{\mathcal{BF}}$ has a countable bipolar fuzzy neighborhood base for every bipolar fuzzy point $c(\gamma, \delta) \in \tau_{\mathcal{BF}}$.

Definition 3.10

A BPVFTS $(X, \tau_{\mathcal{BF}})$ is said to satisfy the second axiom of countability or is said to be a bipolar fuzzy C_{II} space if and only if $\tau_{\mathcal{BF}}$ has a countable bipolar fuzzy base for every bipolar fuzzy point $c(\gamma, \delta) \in \tau_{\mathcal{BF}}$.

Definition 3.11

A family $\{A_\alpha : \alpha \in J\}$ of bipolar fuzzy sets is called a bipolar fuzzy cover of B if and only if $\bigcup_{\alpha\in J}\{A : A \in A_\alpha\} \supseteq B$. It is a bipolar fuzzy open cover if and only if every member of A_α is a bipolar fuzzy open set. If there exists a subset J_1 of J $\ni \bigcup\{A_\alpha : \alpha \in J_1\} \supseteq B,$ then $\{A_\alpha : \alpha \in J_1\}$ is called a bipolar fuzzy subcover.

Definition 3.12

A bipolar valued fuzzy topological space $(X, \tau_{\mathcal{BF}})$ is called a bipolar fuzzy compact space if every bipolar fuzzy open cover for $(X, \tau_{\mathcal{BF}})$ has a finite bipolar fuzzy sub cover.

Definition 3.13

A Bipolar fuzzy topological space $(X, \tau_{\mathcal{BF}})$ is said to be bipolar fuzzy T_2 if for each pair of distinct bipolar fuzzy points $c(\gamma,\delta)$ and $d(\alpha,\beta)$ in X, there exist bipolar fuzzy open neighborhood U of $c(\gamma,\delta)$ and a bipolar fuzzy open neighborhood of V of $d(\alpha,\beta)$ such that U $\bigcap$ V = $O_{+_\sim}$.

Definition 3.14

A Bipolar fuzzy topological space $(X, \tau_{\mathcal{BF}})$ is said to be bipolar fuzzy regular if for each bipolar fuzzy point $c(\gamma,\delta)$ in X and for each bipolar fuzzy closed set F in X not containing $c(\gamma,\delta)$, there exist bipolar fuzzy open neighborhood U of $c(\gamma,\delta)$ and a bipolar fuzzy open neighborhood of V of F such that U $\bigcap$ V = $O_{+_\sim}$. Bipolar fuzzy regular T_1-space is called a bipolar fuzzy T_3-space.

Notation

(i) $I_{0+}{}^{A} \times I_{0-}{}^{A} = \mathcal{B}_\circ^{A}$

Definition 3.15

Let $(X, \tau_{\mathcal{BF}})$ be a bipolar fuzzy topological space. Then

a) $\mathcal{B}_\circ^{X} = \{ F : F \neq 0_{+_\sim}$ And $F \in \tau_{\mathcal{BF}}{}' \}$.

b) $\mathcal{B}_\circ^{A} = \{\{F \in \mathcal{B}_\circ^{X} : F \subset A\}$ and where $A \in \mathcal{B}^{X}$ }.

Notation

$$f(x,y) = \begin{cases} (\alpha,\beta) \; if \; (x,y) = (a,b) \\ (0,0) \; if \; (x,y) \neq (a,b) \end{cases} = d(\alpha,\beta)$$

Definition 3.16

a) Let $(X, \tau_{\mathcal{BF}})$ be a bipolar valued fuzzy topological space is said to be T_0 if for any two distinct bipolar fuzzy points $c(\gamma,\delta)$ and $d(\alpha,\beta)$ where $c, d \in G$

Case 1:

If $(x,y) \neq (u,v)$ either bipolar fuzzy point $c(\gamma,\delta)$ has a bipolar fuzzy open neighborhood, which is not q-coincident with $d(\alpha,\beta)$ or $d(\alpha,\beta)$ has a bipolar fuzzy open neighborhood which is not q-coincident with $c(\gamma,\delta)$.

Case 2:

If $(x, y) = (u, v)$ and $c(\gamma, \delta) < d(\alpha, \beta)$ (i.e) $\gamma < \alpha, \delta < \beta$ then there exists a Q-bipolar fuzzy nbd D of $d(\alpha, \beta)$ which is not q-coincident with $c(\gamma, \delta)$.

b) Let $(X, \tau_{\mathcal{BF}})$ be a bipolar valued fuzzy topological space is said to be T_1 if for any two distinct bipolar fuzzy points $c(\gamma, \delta)$ and $d(\alpha, \beta)$ where $c, d \in G$.

Case 1:

If $(x, y) \neq (u, v)$, $c(\gamma, \delta)$ has a bipolar fuzzy open neighborhood, which is not q-coincident with $d(\alpha, \beta)$ or $d(\alpha, \beta)$ has a bipolar fuzzy open neighborhood which is not q-coincident with $c(\gamma, \delta)$.

Case 2:

If $(x, y) = (u, v)$ and $c(\gamma, \delta) < d(\alpha, \beta)$ (i.e) $\gamma < \alpha, \delta < \beta$, then there exists a Q-bipolar fuzzy nbd D of $d(\alpha, \beta)$ which is q-coincident with $c(\gamma, \delta)$.

Proposition 3.1

A Bipolar fuzzy topological space $(X, \tau_{\mathcal{BF}})$ is T_0 if and only if for any pair of distinct bipolar fuzzy points $c(\gamma, \delta)$ and $d(\alpha, \beta)$, either $c(\gamma, \delta) \notin \{d(\alpha, \beta)\}$ or $d(\alpha, \beta) \notin \{c(\gamma, \delta)\}$.

Proof: By using definition 3.16, it is easy.

Proposition 3.2

$A \subseteq B$ if and only if $A\bar{q}B^C$, particular bipolar valued fuzzy point $c(\gamma, \delta) \subset A$ if and only if $c(\gamma, \delta)\ \bar{q}\ A^C$.

Proof:

To Prove $A\bar{q}\ B^C$ if and only if $A \subseteq B$. Since, by definition $A\bar{q}B^C$ this implies such that $\mu_A^P(x) + 1 - \mu_B^P(x) < 1$ and $\mu_A^N(x) - 1 - \mu_B^N(x) > -1$

$\Leftrightarrow \mu_A^P(x) - \mu_B^P(x) < 0$ and $\mu_A^N(x) - \mu_B^N(x) > 0$

$\Leftrightarrow \mu_A^P(x) < \mu_B^P(x)$ and $\mu_A^N(x) > \mu_B^N(x)$

$\Leftrightarrow A \subseteq B$.

To show that $c(\gamma, \delta) \subset A$ if and only if $c(\gamma, \delta)\ \bar{q}\ A^C$.

Assume that $c(\gamma, \delta)\ \bar{q}$ (i.e) $(c_\gamma, c_\delta)\bar{q}A$. Since, by definition we have

$c_\gamma + 1 - \mu_A^P(\gamma) > 1$ and $c_\delta + -1 - \mu_A^N(\delta) < -1 \Leftrightarrow c_\gamma - \mu_A^P(\gamma) > 0$ and $c_\delta - \mu_A^N(\delta) < 0$

$\Leftrightarrow c_\gamma > \mu_A^P(\gamma)$ and $c_\delta < \mu_A^N(\delta) \Leftrightarrow c(\gamma, \delta) \subset A$

Lemma 3.1

Let $(X, \tau_{\mathcal{BF}})$ be a BPFTS and $\theta \in \mathcal{B}^X$. Then

i) $c(\gamma, \delta) \in BFInt(\theta)$ if and only if $c(\gamma, \delta)$ has a bipolar fuzzy neighborhood contained in θ.

ii) $c(\gamma, \delta) \in BFCl(\theta)$ if and only if for each Q-bipolar fuzzy neighborhood ξ of $c(\gamma, \delta)$ is $\xi q \theta$.

Proof:

(i) It is easy from definition 3.6

(ii) A Bipolar fuzzy point $c(\gamma, \delta) \in BFCl(\theta)$ if and only if, for each bipolar fuzzy closed set $\eta \supset \theta$, $c_\gamma \in \mu_\theta^P(\gamma)$ and $c_\delta \in \mu_\theta^P(\delta)$ (i.e) $\mu_\theta^P(\gamma) \geq \gamma$, $\mu_\theta^N(\delta) \leq \delta$. By taking complement, for the above fact $c(\gamma, \delta) \in BFCl(\theta)$ if and only if, for each bipolar fuzzy open set $\xi \subset \theta^C$, $\mu_\xi^P(\gamma) \leq 1 - \mu_\theta^P(\gamma)$ and $\mu_\xi^N(\delta) \geq -1 - \mu_\theta^N(\delta)$, that is, for every bipolar fuzzy open set ξ satisfies $\xi(\gamma, \delta) > 1 - c(\gamma, \delta)$ (i.e) $\mu_\xi^P(\gamma) > 1 - c_\gamma$ and $\mu_\xi^N(\delta) < -1 - c_\delta$. This implies $\xi(\gamma, \delta)$ is not contained in θ^C. By Proposition 3.2 , $\xi \not\subset \theta^C$ if and only if it is bipolar fuzzy quasi-coincident with $(\theta^C)^c = \theta$.Thus we proved that $c(\gamma, \delta) \in BFCl(\theta)$ if and only if for each Q-bipolar fuzzy neighborhood ξ of $c(\gamma, \delta)$ is $\xi q \theta$. Hence, proved.

Result 3.3

A bipolar valued fuzzy topological space $(X, \tau_{\mathcal{BF}})$ is bipolar fuzzy T_1 space if and only if every singleton set is bipolar fuzzy closed in X.

Proposition 3.3

Let $(X, \tau_{\mathcal{BF}})$ be a bipolar valued fuzzy T_1 space. Then

a) $\mathcal{B}_\circ^{A_0 \cap A_1} = \mathcal{B}_\circ^{A_0} \bigcap \mathcal{B}_\circ^{A_1}$ and $\mathcal{B}_\circ^{\bigcap_\alpha A_\alpha} = \bigcap_\alpha \mathcal{B}_\circ^{A_\alpha}$ where $A_0, A_1, ...A_\alpha \in \mathcal{B}^X$.

b) $A \subset B$ iff $\mathcal{B}_\circ^A \subset \mathcal{B}_\circ^B$ and hence $A = B$ if and only if $\mathcal{B}_\circ^A = \mathcal{B}_\circ^B$ where $A, B \in \mathcal{B}^X$.

Proof:

a) Let $F \in \mathcal{B}_\circ^{A_0 \cap A_1} \Leftrightarrow F \in \mathcal{B}^X \ni F \subset A_0 \bigcap A_1$.

$\Leftrightarrow F \in \mathcal{B}_\circ^X \ni F(x) \leq Min\{\mu_{A_0}^P(x), \mu_{A_1}^P(x)\}$ and $Max\{\mu_{A_0}^N(x), \mu_{A_1}^N(x)\}, \forall x \in X$.

$\Leftrightarrow F \in \mathcal{B}_\circ^X \ni F \subset A_0 \text{ and } F \in \mathcal{B}_\circ^X \ni F \subset A_1$

$\Leftrightarrow F \in \mathcal{B}_\circ^{A_0} \bigcap \mathcal{B}_\circ^{A_1}$

Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a sub collection of $\mathcal{B}_\circ^X$ then $F \in \mathcal{B}_\circ^{\bigcap_{\alpha \in \Lambda} A_\alpha} \Leftrightarrow F \in \mathcal{B}_\circ^X \ni F \subseteq \bigcap_{\alpha \in \Lambda} A_\alpha$

$\Leftrightarrow F \in \mathcal{B}_\circ^X \ni F(x) \subseteq \bigcap_{\alpha \in \Lambda} A_\alpha(x) = \{x, \mu_{\bigcap_{\alpha \in \Lambda} A_\alpha}(x) : x \in X\}$ and $\mu_{\bigcap_{\alpha \in \Lambda} A_\alpha}(x) = (\mu^P_{\bigcap_{\alpha \in \Lambda} A_\alpha}(x), \mu^N_{\bigcap_{\alpha \in \Lambda} A_\alpha}(x)), \forall x \in X$.

where $\mu^P_{\bigcap_{\alpha \in \Lambda} A_\alpha}(x) = Min\{\mu_{A_1}^P(x), \mu_{A_2}^P(x), ...\}, \mu^N_{\bigcap_{\alpha \in \Lambda} A_\alpha}(x) = Max\{\mu_{A_1}^N(x), \mu_{A_2}^N(x), ...\}$

$\Leftrightarrow F \in \mathcal{B}_\circ^X \ni F(x) \subseteq A_\alpha(x), \forall x \in X, \alpha \in \Lambda$.

$\Leftrightarrow F \in \mathcal{B}_{\circ}^{A_\alpha}$ for each $\alpha \in \Lambda$.

$\Leftrightarrow F \in \bigcap_{\alpha\in\Lambda} \mathcal{B}_{\circ}^{A_\alpha} = \bigcap_{\alpha\in\Lambda} \mathcal{B}_{\circ}^{A_\alpha}$.

b) Assume that $\mathcal{B}_{\circ}^{A} \subset \mathcal{B}_{\circ}^{B}$ using definition 3.16, $\Rightarrow \{F \in \mathcal{B}_{\circ}^{X} \ni F \subset A\} \subset \{F \in \mathcal{B}_{\circ}^{X} \ni F \subset B\}$ and where $A, B \in \mathcal{B}^X$. Hence, $A \subset B$. Conversely, suppose that $\mathcal{B}_{\circ}^{A} \subset \mathcal{B}_{\circ}^{B}$ and let $c(\gamma,\delta) \in A$. Since, X is bipolar fuzzy T_1, by Result 3.3 "A bipolar fuzzy topological space $(X, \tau_{\mathcal{BF}})$ is bipolar fuzzy T_1 space if and if every singleton bipolar fuzzy set is closed in X" we have $\{c(\gamma,\delta)\} \in \mathcal{B}_{\circ}^{X}$ and $\{c(\gamma,\delta)\} \subset A$. Thus, $\{c(\gamma,\delta)\} \in \mathcal{B}_{\circ}^{A}$ by the hypothesis $\{c(\gamma,\delta)\} \in \mathcal{B}_{\circ}^{B}$ and hence, $\{c(\gamma,\delta)\} \subset B$. So, $\{c(\gamma,\delta)\} \in B$. Hence, by the Result 3.1, we have $A \subset B$.

Definition 3.17

Let $(X, \tau_{\mathcal{BF}})$ be a BPVFTS and let $\mathcal{A} \in \mathcal{B}^X$. Then $\mathcal{B}_{\circ}^{X} - \mathcal{B}_{\circ}^{A^c} = \{F \in \mathcal{B}_{\circ}^{X} : FqA\}$.

Lemma 3.2

Let $(X, \tau_{\mathcal{BF}})$ be a BPVFTS and let $\Im$ be a collection of all sets $\mathcal{B}_{\circ}^{G}$ and of all sets $\mathcal{B}_{\circ}^{X} - \mathcal{B}_{\circ}^{G^C}$ where $G \in \tau_{\mathcal{BF}}$. Let $\mathcal{B}_{\mathcal{F}e} = \{\bigcap_{i\in J} \Im\}$ be the collection of all finite intersection of bipolar fuzzy members of $\Im$. Then for all $\mathbf{B} \in \mathcal{B}_{\mathcal{F}e}$, $\mathbf{B} = \mathcal{B}_{\circ}^{X} - \mathcal{B}_{\circ}^{A^c} = \{F \in \mathcal{B}_{\circ}^{X} : F \subset A_{\circ}$ and FqA_i for each $i = 1,2,....n\}$, where $A_i \in \tau_{\mathcal{BF}_\mathcal{V}}$, for each $i = 0,1,2,....n$ and here, $\mathbf{B}$ is denoted as $\langle A_0, A_1, ...A_n \rangle_{\mathcal{F}e}$.

Proof:

Let us consider $\mathbf{B} \in \mathcal{B}_{\mathcal{F}e}$, then $\exists$ bipolar fuzzy open sets $A_0, A_1, ...A_n$ in X such that $\mathbf{B} = \mathcal{B}_{\circ}^{A_\circ} \bigcap (\mathcal{B}_{\circ}^{X} - \mathcal{B}_{\circ}^{A_1^c}) \bigcap .... \bigcap (\mathcal{B}_{\circ}^{X} - \mathcal{B}_{\circ}^{A_n^c}) = \{\eta \in \mathcal{B}_{\circ}^{X} : \eta \subset A_{\circ}, \eta q A_i$ for each $i = 1,2,....n\}$. and by Definition 3.17 $\mathbf{B} = \{\{F \in \mathcal{B}_{\circ}^{X} : F \subset A_{\circ}\} \bigcap \{F \in \mathcal{B}_{\circ}^{X} : FqA_1\} \bigcap ... \bigcap \{F \in \mathcal{B}_{\circ}^{X} : FqA_n\}$, for each $i = 1,2,....n\}$. Thus, $\mathbf{B} = \{F \in \mathcal{B}_{\circ}^{X} : F \subset A_{\circ}$ and FqA_i for each $i = 1,2,....n\}$.

Proposition 3.4

Let $(X, \tau_{\mathcal{BF}})$ be a BPVFTS and let $\Im$ be a collection of all sets $\mathcal{B}_{\circ}^{G}$ and of all sets $\mathcal{B}_{\circ}^{X} - \mathcal{B}_{\circ}^{G^c}$ where $G \in \tau_{\mathcal{BF}}$. Then there is a unique bipolar fuzzy topology $\tau_{\mathcal{BF}_e}$ (called Bipolar fuzzy exponential topology) on $\mathcal{B}_{\circ}^{X}$ such that $\Im$ is a bipolar fuzzy subbase for $\tau_{\mathcal{BF}}$ and hence $\mathcal{BF}_e$ is a bipolar fuzzy base for $\tau_{\mathcal{BF}_e}$.

Proof: By using Lemma 3.2, we obtain this result easily.

Definition 3.18

Let $(X, \tau_{\mathcal{BF}})$ be a bipolar fuzzy topological space. Then the bipolar fuzzy vietories (finite) topology $\tau_{\mathcal{BF}_\mathcal{V}}$ on $\mathcal{B}_{\circ}^{X}$ is the generated by the collection of the forms $\langle V_1, V_2, V_3, ..., V_n \rangle_{\mathcal{BF}_\mathcal{V}}$ with $V_1, V_2, V_3, ..., V_n$ bipolar fuzzy open sets in $(X, \tau_{\mathcal{BF}})$, where $\left\langle V_1,V_2,V_3,....,V_n \right\rangle_{\mathcal{BF}_{\mathcal{V}}} = \{F \in \mathcal{B}_{\circ}{}^X : F \subset \bigcup_{i=1}^{n} V_i$ and FqV_i for each i = 1,2….n}. The pair $(\mathcal{B}_{\circ}{}^X, \tau_{\mathcal{BF}_{\mathcal{V}}})$ is called a Bipolar fuzzy vietories topological space (Bipolar fuzzy hyperspace).

Proposition 3.5

The collection ζ, $\mathcal{BF}_{\mathcal{V}}$ of the forms $\left\langle V_1,V_2,V_3,....,V_n \right\rangle_{\mathcal{BF}_{\mathcal{V}}} = \{F \in \mathcal{B}^X : F \subset \bigcup_{i=1}^{n} V_i$ and FqV_i for each i = 1,2….n } with $V_1,V_2,V_3,....,V_n$ are bipolar fuzzy open sets in X forms a bipolar fuzzy base for $\tau_{\mathcal{BF}_{\mathcal{V}}}$.

Proof:

To prove the collection ζ, $\mathcal{BF}_{\mathcal{V}}$ forms a bipolar fuzzy base for $\tau_{\mathcal{BF}_{\mathcal{V}}}$ Since, $\left\langle \mathcal{B}_{\circ}{}^X \right\rangle = \left\langle X \right\rangle$ and $\left\langle X \right\rangle \in \mathcal{BF}_{\mathcal{V}}$, $\mathcal{B}_{\circ}{}^X = \bigcup \mathcal{BF}_{\mathcal{V}}$. Let $\mathcal{V} = \left\langle V_1,....V_n \right\rangle_{\mathcal{BF}_{\mathcal{V}}}$, $\mathcal{W} = \left\langle W_1,....W_n \right\rangle_{\mathcal{BF}_{\mathcal{V}}}$, $\mathcal{V} = \bigcup_{i=1}^{m} V_i$ and $\mathcal{W} = \bigcup_{j=1}^{m} W_i$, then clearly V and $W \in \tau_{\mathcal{BF}_{\mathcal{V}}}$. Consider $F \in \left\langle V_1 \cap \mathcal{W},....V_n \cap \mathcal{W}, W_1 \cap \mathcal{V},....W_n \cap \mathcal{V} \right\rangle_{\mathcal{BF}_{\mathcal{V}}}$. Then $F \subset \left[\bigcup_{i=1}^{n}(V_i \cap \mathcal{W})\right] \cup \left[\bigcup_{j=1}^{m}(W_i \cap \mathcal{V})\right]$, $Fq\left[(V_i \cap \mathcal{W})\right]$ and $Fq\left[(W_i \cap \mathcal{V})\right]$ for each i = 1,…n and j = 1,…m. Thus, $F \subset \left[\mathcal{V} \cap \mathcal{W}\right]$, (i.e) $F \subset \mathcal{V}$ and $F \subset \mathcal{W}$, $F \subset V_i$ $F \subset W_j$ for each i = 1,…n and j = 1,…m. So $F \in \mathcal{V} \cap \mathcal{W}$.

Proposition 3.6

Let $\mathcal{BF}_e$ and $\mathcal{BF}_{\mathcal{V}}$ are equivalent (i. e) bipolar fuzzy exponential topology and bipolar fuzzy vietories topology are also equal

Proof:

Let $\left\langle G_0, G_1,....G_n \right\rangle_{\mathcal{BF}_e} \in \mathcal{B}_{\mathcal{F}_e}$ and let $F \in \left\langle G_0, G_1....G_n \right\rangle_{\mathcal{BF}_e}$. Then also $F \subset G_0$ implies $F < G_0$ and FqG_i for each i = 1, 2,…n. Let $A_i = G_0 \bigcup G_i$ for each i = 1,2,…n. (i.e) $A_i = G_0 \bigcup G_i = \left\langle x, G_0 \vee G_i, G_0 \wedge G_i \right\rangle$. Then A_i is a bipolar fuzzy open set in X for each i = 1, 2,…n and thus $F \in \left\langle A_1....A_n \right\rangle_{\mathcal{BF}_v}$. Let $F \in \left\langle A_1,....A_n \right\rangle_{\mathcal{BF}_v}$. Then $E \subset \bigcup_{i=1}^{n} A_i$ and EqA_i for each i = 1, 2, …n. Let $G_0 = \bigcup_{i=1}^{n} A_i$ and $A_i \cap G_0 = G_i$ for each i = 1, 2,…n. Then G_i is bipolar fuzzy open in X for each i = 1,2,…n, $E \in \left\langle G_0, G_1,....G_n \right\rangle_{\mathcal{BF}_e}$. Similarly, for each $\left\langle A_0, A_1....A_n \right\rangle_{\mathcal{BF}_v} \in \mathcal{BF}_V$ and each $F \in \left\langle A_1,....A_n \right\rangle_{\mathcal{BF}_v}$ there exists a $\left\langle G_0, G_1,....G_n \right\rangle_{\mathcal{BF}_e} \in \mathcal{BF}_V$ such that $E \in \left\langle G_0, G_1....G_n \right\rangle_{\mathcal{BF}_e} \subset \left\langle A_1....A_n \right\rangle_{\mathcal{BF}_v}$. Thus, $\mathcal{BF}_e$ and $\mathcal{BF}_{\mathcal{V}}$ are equivalent.

Lemma 3.3

Let $(X, \tau_{\mathcal{BF}})$ be a bipolar fuzzy T_1 Space and $A \in \mathcal{B}^X$. Then

a) $\overline{\mathcal{B}_{\circ}^{A}} \subset \mathcal{B}_{\circ}^{\bar{A}}$ but if A(x) < (1/2, -1/2) for each $x \in X$ then $\overline{\mathcal{B}_{\circ}^{A}} \subset \mathcal{B}_{\circ}^{\bar{A}}$.

b) $\overline{\mathcal{B}_{\circ}^{\mathring{A}}} \subset \mathcal{B}_{\circ}^{\mathring{A}}$.

Proof:

Since, by Proposition 3.3 (b) $\mathcal{B}_\circ^{A} \subset \mathcal{B}_\circ^{\bar{A}}$, $\mathcal{B}_\circ^{A} = \langle x, \mu_A^{P}, \mu_A^{N} \rangle \subset \mathcal{B}_\circ^{\bar{A}} = \langle x, \mu_A^{N}, \mu_A^{P} \rangle$.
$\mathcal{B}_\circ^{X} - \mathcal{B}_\circ^{\bar{A}} = \mathcal{B}_\circ^{X} - \mathcal{B}_\circ^{\left(\overset{\circ}{A^C}\right)^C} = \{ F \in \mathcal{B}_\circ^{X} : Fq\left(\overset{\circ}{A^C}\right)^c \}$ by the Result 3.2 and Definition 3.17. Hence, by Lemma 3.2 and Proposition 3.3, $\mathcal{B}_\circ^{X} - \mathcal{B}_\circ^{\bar{A}}$ is bipolar fuzzy open set in $(\mathcal{B}_\circ^{X}, \tau_{\mathcal{BF}_\mathcal{V}})$. So, $\mathcal{B}_\circ^{\bar{A}}$ is bipolar fuzzy closed set in $(\mathcal{B}_\circ^{X}, \tau_{\mathcal{BF}_\mathcal{V}})$ and thus $\overline{\mathcal{B}_\circ^{A}} \subset \mathcal{B}_\circ^{\bar{A}}$. Let $F(\gamma,\delta) \in \mathcal{B}_\circ^{\bar{A}}, \mu_{\mathcal{B}_\circ^{\bar{A}}}{}^{P}(x) \geq \gamma,\ \mu_{\mathcal{B}_\circ^{\bar{A}}}{}^{N}(x) \leq \delta$ (i.e) $F \subset \bar{A}$. Therefore, $F(\gamma,\delta) \subset \bar{A}$. Let $\langle G_1, ...G_n \rangle_{\mathcal{BF}_\mathcal{V}}$ be any base for the bipolar fuzzy member of $\tau_{\mathcal{BF}_\mathcal{V}}$ containing F. Then $F \subset \bigcup_{i=1}^{n} G_i$ and FqG_i for each i = 1, 2,…n. Hence, there is an $x_i \in X \ni FqG_i$ if $\mu_{G_i}{}^{P}(x_i) + \gamma_F > 1$ and $\mu_{G_i}{}^{N}(x_i) + \delta_F < -1$ (i.e) $x_i(\gamma,\delta) = \langle x, x_{i_\gamma}, x_{i_\delta} \rangle$.

Let $F(x_i) = H$ for each i = 1,2,…n. Then, $H_i qG_i$ and $\mu_{G_i}{}^{P}(x_i) + H_i > 1$ and $\mu_{G_i}{}^{N}(x_i) + H_i < -1$, Since, $F(x_i) = H$ and thus $x_i(\gamma_i,\delta_i)qG_i$ for each i = 1,…n. Since, $x_i(\gamma_i,\delta_i) \in \bar{A}, AqG_i$ by Lemma 3.1 (b), thus there exists $y_i \in X \ni A(y_i)qG_i(y_i)$ for each i = 1,2…n, $\mu_{G_i}{}^{P}(y_i) + \mu_A{}^{P}(y_i) > 1$ and $\mu_{G_i}{}^{N}(y_i) + \mu_A{}^{N}(y_i) < -1$. Let $A(y_i) = \lambda_i$ for each i = 1,2…n and let $C = \{x_1(\gamma_1,\delta_1)_{\lambda_1}, x_2(\gamma_2,\delta_2)_{\lambda_2} ....x_n(\gamma_n,\delta_n)_{\lambda n})\}$. Then $C \in \mathcal{B}_\circ^{X}, C \subset A$ and CqG_i for each i = 1,2…n. Conversely, since, A(x) < $(1/2, -1/2)$ for $x \in X$, $A(y_i) = \lambda_i < (1/2, -1/2)$ for each i = 1,2…n. Hence, $G_i(y_i) > (1/2, -1/2) > \lambda_i$ for each i = 1,2…n. So $C \subset \bigcup_{i=1}^{n} G_i$ and Thus $C \in B_\circ^{A} \bigcap \langle G_1....G_n \rangle_{\mathcal{BF}_\mathcal{V}} \neq \phi, C \subset \bigcup_{i=1}^{n} G_i$ and $C \in \overline{B^A}$. Hence $C \in \overline{B^A}$ (i.e) $\overline{B^A} \supset B^{\bar{A}}$. Therefore, $\overline{B^A} = B^{\bar{A}}$.

b) Since, by Lemma 3.2 and Proposition 3.6 $\mathcal{B}_\circ^{\overset{\circ}{A}}$ is bipolar fuzzy open in $(\mathcal{B}_\circ^{X}, \tau_{\mathcal{BF}_\mathcal{V}})$, by Proposition 3.3 b) $\mathcal{B}_\circ^{\overset{\circ}{A}} \subset \mathcal{B}_\circ^{A}$ so $\mathcal{B}_\circ^{\overset{\circ}{A}} \subset \overset{\circ}{\overline{\mathcal{B}_\circ^{A}}}$. Now let $F \notin \mathcal{B}_\circ^{\overset{\circ}{A}}$. Then by the Result 3.2, $F = \left(\overline{A^c}\right)^c$. Hence, by the Definition 2.2, $Fq\overline{A^c}$. So there is a $y \in X \ni F(y)q\overline{A^c}(y)$ This implies $\mu_{\overline{A^c}}{}^{P}(y) + \mu_F{}^{P}(y) > 1$, $\mu_{\overline{A^c}}{}^{N}(y) + \mu_F{}^{N}(y) < -1$. Let $\overline{A^c}(y) = \rho$. Then $y(\gamma_\rho,\delta_\rho) \in \left(\overline{A^c}\right)$ and $y(\gamma_\rho,\delta_\rho)qF$. Let $\langle G_0, G_1....G_n \rangle_{\mathcal{BF}_{\mathcal{V}_e}}$ be any bipolar fuzzy member base for $\tau_{\mathcal{BF}_{\mathcal{V}_e}}$ containing F. Then $F \subset \bigcup_{i=1}^{n} G_i$ (i.e) $F \subset G_0$ and FqG_i for each i = 1,2,…n. Hence, $y(\gamma_\rho,\delta_\rho)qG_0$ implies $\mu_{y_{\gamma_\rho}}{}^{P}(x) + \mu_{G_0}{}^{P}(x) > 1$, $\mu_{y_{\delta_\rho}}{}^{N}(x) + \mu_{G_0}{}^{N}(x) < -1$. Since, $y(\gamma_\rho,\delta_\rho) \in \left(\overline{A^c}\right)$ by Lemma 3.1 (b), $A^c qG_0$ so there is a $x \in X \ni A^c(x) + G_0(x) > 1$. Let $G_0(x) = \lambda$. Then clearly $c(\gamma,\delta) \in G_0$ and $c(\gamma,\delta)qA^c$. Let $C = F \bigcup \{c(\gamma,\delta)\}$. Then $F \in \mathcal{B}_\circ^{X}$, FqA^c and $F \in \mathcal{B}_\circ^{X} - \mathcal{B}_\circ^{A}$, also $C \subset G_0$ and CqG_i for each i = 1,2,…n. So, $C \in \langle G_0, G_1....G_n \rangle_{\mathcal{BF}_{\mathcal{V}_e}} \bigcap (I_0{}^{X} - I_0{}^{A}) \neq \phi$, $C \in \{\langle G_0, G_1...G_n \rangle_{\mathcal{BF}_{\mathcal{V}_e}}\ q(\mathcal{B}_0^{X} - \mathcal{B}_0^{A})\}$. Thus $C \in \overline{\mathcal{B}_\circ^{X} - \mathcal{B}_\circ^{A}}$. Hence, $F \notin \overset{\circ}{\overline{\mathcal{B}_\circ^{A}}}$ $F \in \overset{\circ}{\overline{\mathcal{B}_\circ^{A}}} \subset \mathcal{B}_\circ^{\overset{\circ}{A}}$. Therefore, $\overset{\circ}{\overline{\mathcal{B}_\circ^{A}}} = \mathcal{B}_\circ^{\overset{\circ}{A}}$.

Lemma 3.4

Let $(X, \tau_{\mathcal{BF}})$ be a bipolar fuzzy T_1 Space and let $A \in \mathcal{B}_\circ^X$.

a) $\mathcal{B}_0^A$ and $\mathcal{B}_0^X - \mathcal{B}_0^{A^C}$ are bipolar fuzzy open in $\mathcal{B}_0^X$ if and only if A is bipolar fuzzy open in X.

b) If A is bipolar fuzzy closed in X, then $\mathcal{B}_0^X$ and $\mathcal{B}_0^X - \mathcal{B}_0^{A^C}$ are bipolar fuzzy closed in $\mathcal{B}_0^X$.

c) If $\mathcal{B}_0^A$ and $\mathcal{B}_0^X - \mathcal{B}_0^{A^C}$ are bipolar fuzzy closed in $\mathcal{B}_0^X$ and A(x) < $(1/2, -1/2)$ for each $x \in X$, then A is bipolar fuzzy closed in X.

Proof: By using Lemma 3.3, proof is simple.

Proposition 3.7

Let $(X, \tau_{\mathcal{BF}})$ be a bipolar fuzzy T_3 Space and let $A \in \mathcal{B}_\circ^X$. Then the set $\{F \in \mathcal{B}_\circ^X \ni A \subset F\}$ is bipolar fuzzy closed in $(\mathcal{B}_0^X, \tau_{\mathcal{BF}_\mathcal{V}})$.

Proof:

Let $\mathcal{A} = \{F \in \mathcal{B}_\circ^X \ni A \subset F\}$. Then $\mathcal{A}^c = \{F \in \mathcal{B}_\circ^X \ni A \not\subset F\}$.

$$= \bigcup_{c(\gamma,\delta)\in A} \{F \in \mathcal{B}_0^X \ni F \subset \{c(\gamma,\delta)\}^c\}.$$

$$= \bigcup_{c(\gamma,\delta)\in A} \{\mathcal{B}_0^{\{c(\gamma,\delta)\}^c}\}.$$

Since, X is bipolar fuzzy T_1 by Result 3. 3 $\{c(\gamma,\delta)\}$ is bipolar fuzzy closed in $(X, \tau_{\mathcal{BF}})$ for each $\{c(\gamma,\delta)\} \in \mathcal{B}_0^X(X)$. Thus, $\{c(\gamma,\delta)\}^c$ is open in $(X, \tau_{\mathcal{BF}})$. So by Lemma 3.4 (a) $\mathcal{B}_0^{\{\{c(\gamma,\delta)\}\}^c}$ is open in $(X, \tau_{\mathcal{BF}})$ and thus $\mathcal{A}^c$ is open in $(X, \tau_{\mathcal{BF}})$. Hence, $\mathcal{A}$ is closed in $(X, \tau_{\mathcal{BF}})$.

Proposition 3.8

Let $(X, \tau_{\mathcal{BF}})$ be a bipolar fuzzy topological space. Then

a) $(\mathcal{B}_0^X, \tau_{\mathcal{BF}_\mathcal{V}})$ is bipolar fuzzy T_0.

b) If X is T_1, then $\mathcal{B}_0^X$ is bipolar fuzzy T_1 and the converse is false.

Proof:

a) Let $A, B \in \mathcal{B}_0^X \ni A \neq B$. Let $c(\gamma,\delta) \in A, c(\gamma,\delta) \notin B$ and $W = B^c$. Then W is open in X, $AqU, AqX, A \subset W \cup X$ and $B\bar{q}W$. Hence $A \in \langle W, X \rangle_{\mathcal{BF}_\mathcal{V}}$ and $B \in \langle W, X \rangle_{\mathcal{BF}_\mathcal{V}}$. Thus, $(\mathcal{B}_0^X, \tau_{\mathcal{BF}_\mathcal{V}})$ is bipolar fuzzy T_0.

b) Let $E \in \mathcal{B}_0^X$. Then $\{E\} = \{F \in \mathcal{B}_0^X \ni F = E\}$

$$= \{F \in \mathcal{B}_0^X \ni F \subset E\} \cap \{F \in \mathcal{B}_0^X \ni E \subset F\}.$$

Thus, by Lemma 3.4 and Theorem 3.7, $\{E\}$ is bipolar fuzzy closed in $\mathcal{B}_0^{\ X}$. Hence $\mathcal{B}_0^{\ X}$ is bipolar fuzzy T_1.

Definition 3.19

A bipolar fuzzy set A in a bipolar valued fuzzy topological space $(X, \tau_{\mathcal{BF}})$ is said to be bipolar fuzzy dense if $BFCl(A) = 1_{\underset{\sim}{+}}$ there exists no bipolar fuzzy closed set B in $(X, \tau_{\mathcal{BF}})$ such that $A \subset B \subset 1_{\underset{\sim}{+}}$. Let A is said to be countably bipolar fuzzy dense in $(X, \tau_{\mathcal{BF}})$ if A is bipolar fuzzy dense in $(X, \tau_{\mathcal{BF}})$ and S(A) is bipolar fuzzy countable. If $(X, \tau_{\mathcal{BF}})$ has a bipolar fuzzy countable dense set, then we say that $(X, \tau_{\mathcal{BF}})$ is bipolar fuzzy separable.

Definition 3.20

A bipolar fuzzy set A in a fuzzy topological space $(X, \tau_{\mathcal{BF}})$ is called bipolar fuzzy nowhere dense if there exists no non-zero bipolar fuzzy open set B in $(X, \tau_{\mathcal{BF}})$ such that B $\subset$ BFCl(A) (i.e) BFInt(BFCl(A)) = $0_{\underset{\sim}{+}}$.

Proposition 3.9

Let $\Im(X)$ be the family of all the bipolar fuzzy finite sets in a bipolar fuzzy T_1 space X. Then $\Im(X)$ is dense in $(\mathcal{B}_0^{\ X}, \tau_{\mathcal{BF}_{\mathcal{V}}})$.

Proof:

Let $F \in \mathcal{B}_0^{\ X}$ and let $\langle G_1,..G_n \rangle_{\mathcal{BF}_{\mathcal{V}}}$ be any bipolar fuzzy finite base member for $\tau_{\mathcal{BF}_{\mathcal{V}}} \ni F \in \langle G_1,..G_n \rangle_{\mathcal{BF}_{\mathcal{V}}}$. Then $F \subset \bigcup_{i=1}^{n} G_i$ and FqG_i for each i = 1,2...n. Let $c_i(\gamma_i, \delta_i)_{C_i} \in F$ and $c_i(\gamma_i, \delta_i)_{C_i}\ qG_i(x_i)$, for each i = 1,2....n. Let $C = \{c_i(\gamma_i, \delta_i)\}$. Then $C \in \Im(X) \bigcap \langle G_1,..G_n \rangle_{\mathcal{BF}_{\mathcal{V}}} \neq \phi$ and thus $F \in \overline{\Im(X)}$, i.e., $\mathcal{B}_0^{\ X} \subset \overline{\Im(X)}$. So, $\overline{\Im(X)} = \mathcal{B}_0^{\ X}$. Hence, $\Im(X)$ is dense in $\mathcal{B}_0^{\ X}$.

Proposition 3.10

If X is bipolar fuzzy separable if and only if $\mathcal{B}_0^{\ X}$ is bipolar fuzzy separable.

Proof:

Suppose X is bipolar fuzzy separable. Let *D* be bipolar fuzzy countable dense set in X and let $\boldsymbol{D}$ be the collection of finite bipolar fuzzy subsets of *D*. Then $\boldsymbol{D}$ is countable. Let $\langle G_1,..G_n \rangle_{\mathcal{BF}_{\mathcal{V}}}$ be a base member for $\tau_{\mathcal{BF}_{\mathcal{V}}}$. Since, *D* is dense in X, by Lemma 3.1(ii), $D\,qG_i$ for each i = 1,..n. Let $c_i(\gamma, \delta)_{\vartheta_i} \in D$ and $H_i qG_i(x_i)$, for each i = 1,..n. Let $F = \{c_1(\gamma, \delta)_{H_1}, ...c_n(\gamma, \delta)_{H_n}\}$. Then, $F \in \boldsymbol{D} \bigcap \langle G_1,..G_n \rangle_{\mathcal{BF}_{\mathcal{V}}}$. Thus $\boldsymbol{D}$ is countable dense in $\mathcal{B}_0^{\ X}$ is bipolar fuzzy separable.

Conversely, suppose that $\mathcal{B}_0^{\ X}$ is bipolar fuzzy separable. Let $\boldsymbol{D} = \{A_n \ni n \in Z_{0+} \times Z_{0-}\}$ be a countable bipolar fuzzy dense subset of $\mathcal{B}_0^{\ X}$ where $Z_{0+} \times Z_{0-}$ is a subset of $\mathcal{B}_0^{\ X}$. Let for every $A_n \in \boldsymbol{D}$, choose a bipolar fuzzy point $c(\gamma,\delta) \in A_n$ and let $D = \{c_n(\gamma,\delta)_{\lambda_n} \ni n \in Z_{0+} \times Z_{0-}\}$. Let W be bipolar fuzzy open set in X. Then $\langle W \rangle_{\mathcal{BF}_{\mathcal{V}}}$ is bipolar fuzzy open in $\mathcal{B}_0^{\ X}$. This implies, there is $A_n \in \boldsymbol{D} = \langle W \rangle_{\mathcal{BF}_{\mathcal{V}}}$ so, $A_n \subset W$ and $A_n qW$ and thus $c_n(\gamma,\delta) \subset W$ and $Wq\,D$. So $D = X$. Hence X is bipolar fuzzy separable.

IV. CONCLUSIONS

In this paper we developed, a bipolar fuzzy vietories topological space is obtained from bipolar fuzzy closed set and bipolar fuzzy quasi coincident. By using the bipolar fuzzy point, we defined the bipolar fuzzy T_0 and T_1 space and investigate the bipolar fuzzy exponential and bipolar fuzzy vietories topology are equivalent. The concept of bipolar fuzzy can be applied in different areas like engineering, computer networking systems, signal processing and medical diagnosis etc. We extend our research to bipolar vague fuzzy vietories topological space and bipolar fuzzy soft vietories topological space.

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