Locally Multiplicatively Convex Algebra

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Abstract

If the algebra is complete and over $\mathbb{C}$, the spectra are always nonempty. I have given an alternative proof of $\sigma_{\mathcal{A}}(x)$ is not empty for locally m-convex Algebra.

key words: Locally m-convex Algebra, Seminorm, division Algebra, Maximal Ideal

Introduction

Since the Memoir [**M**] of Michael's Thesis, the theory of locally multiplicatively convex topological algebras has attracted many researchers. This is addressed to prepare a comfortable platform to self study the Memoir. I investigate locally multiplicatively convex algebras and give various examples of such algebras. I investigate various aspects and concepts related with such algebras, spectrum is one such concept and the Gel'fand-Mazur theorem is one such aspect. Unlike Banach algebras, spectrum of an element of an lmc algebra may be unbounded.

0.1. Definition. [**M**, p. 6] A *topological algebra* is an associative algebra $\mathcal{A}$ which is also a topological vector space and such multiplication is a continuous function from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. That is the map

$$(x, y) \in \mathcal{A} \times \mathcal{A} \mapsto xy \in \mathcal{A}$$

is continuous when $\mathcal{A} \times \mathcal{A}$ is equipped with the product topology.

Thus when $x, y \in \mathcal{A}$ and $V \in \mathscr{T}$ is an open neighborhood V of xy there must be open neighborhoods $V_x, V_y \in \mathscr{T}$ of x and y respectively such that

$$V_x V_y \subseteq V,$$

where $V_x V_y = \{xy : x \in V_x, y \in V_y\}$.

1. Idempotent And Multiplicatively Convex Sets

1.1. Definitions. [**M**, Definition 1.1, p.8] Let $\mathcal{A}$ be topological algebra. A subset U of $\mathcal{A}$ is called *idempotent* if $UU \subset U$ where $UU = \{xy : x, y \in U\}$; it is called *m-convex (or multiplicatively convex)* if it is convex and idempotent.

Let U is a subset of the algebra $\mathcal{A}$ which intersects each real, one-dimensional subspace in a proper, closed interval about 0, that is there exists $r_1 < 0 < r_2$ such that $\{\lambda x : \lambda \in \mathbb{R}\} \cap U = [r_1, r_2]x$. Then we define the Minkowski functional of U to be

$$p_U(x) = \inf\{\lambda > 0 : x \in \lambda U\}, (x \in \mathcal{A}). \tag{1.1.1}$$

1.2. Proposition. *Let $\mathcal{A}$ be a topological algebra. U and p_U be as in the last paragraph. Then $U = \{x \in \mathcal{A} : p_U(x) \leq 1\}$.*

2. Locally Multiplicatively Convex Algebras

2.1. Definition. [**M**, Definition 2.1, p. 9] A topological algebra (resp. linear space) is called *locally m-convex (resp. convex)* if there exists a basis for the neighborhoods of the origin consisting of sets which are m-convex(resp. convex) symmetric.

Equivalently, a topological algebra $\mathcal{A}$ is *locally m-convex* iff its topology is generated by a set $\{p_i\}_{i\in I}$ of seminorms, each satisfying $p_i(xy) \leq p_i(x)p_i(y)$. Such seminorms are called submultiplicative.

2.2. Theorem. [**Ru**, Theorem 1.37, p. 27] *Suppose $\mathcal{P}$ is a separating family of seminorms on a vector space $\mathcal{A}$. Associate to each $p \in \mathcal{P}$ and to each positive integer n the set*

$$V(p, n) = \left\{x : p(x) < \frac{1}{n}\right\}.$$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(p, n)$. Then $\mathscr{B}$ is a convex symmetric local base for a topology $\mathscr{T}$ on $\mathcal{A}$, which turns $\mathcal{A}$ into a locally convex space such that every $p \in \mathcal{P}$ is continuous.

The following are all examples of locally m-convex algebras.

2.3. Example. The space $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : f \text{ is continuous}\}$ can't be "normed" with a sup-norm over all of $\mathbb{R}$ (a continuous function on $\mathbb{R}$ could be unbounded), but it can be made into a locally m-convex algebra as follows.

$\mathbb{R}$ is the union of countably many compact sets $[-n, n]$ which can be choose so that $[-n, n]$ lies in the interior of $[-n-1, n+1]$ where $n = 1, 2, \ldots$. $C(\mathbb{R})$ is the vector space of all complex-valued continuous functions on $\mathbb{R}$, topology by the separating family of seminorms

$$p_n(f) = \sup_{x\in[-n,n]} |f(x)|$$

in accordance with Theorem 2.2. Since $p_1 \leq p_2 \leq \cdots$, the sets

$$V_n = \left\{ f \in C(\mathbb{R}) : p_n(f) < \frac{1}{n} \right\} \quad (n = 1, 2, \ldots)$$

form a convex local base for $C(\mathbb{R})$.

2.4. Example. Let $S = \{z \in \mathbb{C} : |Re(z)| \leq 1\}$. The space

$$C(S) = \{f : S \to \mathbb{C} : f \text{ is continuous}\}.$$

Choose compact sets K_n $(n = 1, 2, \ldots)$ such that K_n lies in the interior of K_{n+1} and $S = \cup K_n$. $C(S)$ is the vector space of all complex valued continuous functions on S, topology by the separating family of seminorms

$$p_n(f) = \sup_{z \in K_n} |f(z)|$$

in accordance with Theorem 2.2. Since $p_1 \leq p_2 \leq \cdots$ the sets

$$V_n = \left\{ f \in C(S) : p_n(f) < \frac{1}{n} \right\} \quad (n = 1, 2, \ldots)$$

form a convex local base for $C(S)$.

2.5. Lemma. [**M**, Lemma 2.5, p. 12] *Let $\mathcal{A}$ be an algebra (res. linear space), and let U be an m-convex (resp. convex) and symmetric set which spans $\mathcal{A}$. Let p be the pseudonorm associated with U, and N_p the null set of p. Then N_p is an ideal (resp. subspace) in $\mathcal{A}$, and $p(x) = p(y)$ whenever $x - y \in N_p$. If we now define $\widetilde{p}$ on $\mathcal{A}/N_p$ $(\mathcal{A}_p)$ by $\widetilde{p}(x + N_p) = p(x)$, then $\widetilde{p}$ is a normed on $\mathcal{A}/N_p$, which makes $\mathcal{A}/N_p$ into a normed algebra (resp. space). If $\overline{\mathcal{A}_p}$ denote the completion of $\mathcal{A}/N_p$ $(\mathcal{A}_p)$ with $\widetilde{p}$, then $\overline{\mathcal{A}_p}$ is a Banach algebra.*

Notations. Let $\mathcal{A}$ be a locally m-convex algebra (resp. locally convex space) and let $\{U_i\}$ be an m-base (resp. base) for $\mathcal{A}$. Then, for each i,

(1) "p_i" denotes the *pseudonorm* on $\mathcal{A}$ generated by U_i.
(2) "N_i" denoted the *null space* of p_i.
(3) "$\widetilde{p_i}$" denotes the *norm induced* by p_i on the abstract algebra (resp. linear space) $\mathcal{A}/N_i$ as in Lemma 2.5.
(4) "$\mathcal{A}_i$" denoted the *normed algebra (resp. space)* which we get by equipping $\mathcal{A}/N_i$ with the norm p_i.
(5) "$\overline{\mathcal{A}_i}$" denotes the *completion* of $\mathcal{A}_i$.
(6) "π_i" denotes the *natural homomorphism* from $\mathcal{A}$ onto $\mathcal{A}_i$.
(7) "x_i" denoted $\pi_i(x) \quad (x \in \mathcal{A})$.

2.6. Remark. Observe that π_i is continuous. Choose any $x \in \mathcal{A}$. Since 0 is one of elements of N_i,

$$\|\pi_i(x)\| = \|x + N_i\| = \inf_{m \in N_i} \|x - m\| \leq \|x - 0\| = \|x\|.$$

Hence $\|\pi_i(x)\| \leq \|x\|$.

2.7. Proposition. [**M**, Proposition 2.7, p. 13] *A topological algebra (resp. linear space) is locally m-convex (resp. convex) if and only if it is isomorphic to a subalgebra (resp. sub-space) of a cartesian product of normed algebras (resp. spaces).*

PROOF. A subalgebra (resp. subspace) of a cartesian product of normed algebras (resp. spaces) is certainly locally m-convex (convex), by Proposition **??**.
Conversely, suppose that $\mathcal{A}$ is a locally m-convex algebra (resp. locally convex space), and let $\{U_i\}$ be m-base (resp. base) for $\mathcal{A}$. Then, using above Notation, it follows from Definition 2.1 that the topology on $\mathcal{A}$ is the weakest topology for which all the π_i are continuous. Define, $\pi : \mathcal{A} \to \prod(\mathcal{A}_i)$ by

$$\pi(x) = \pi_i(x) = x_i.$$

Then clearly π is well defined and continuous also isomorphism. $\pi(\mathcal{A})$ is a subalgebra which is isomorphic to the subalgebra of cartesian product of normed algebra. □

2.8. Definition. [**M**] Let $\mathcal{A}$ be an algebra, then an element $x \in \mathcal{A}$ is called *quasi-regular (q.r.)* if there exists an elements $y \in \mathcal{A}$ such that $y \circ x = x \circ y = 0$. (Here $x \circ y = x + y - xy$.)

2.9. Definition. [**M**, p. 7] If x is an element of the algebra $\mathcal{A}$, the $\sigma_{\mathcal{A}}(x)$ denotes the spectrum of x in $\mathcal{A}$. It is denoted as

$$\sigma_{\mathcal{A}}(x) = \{\lambda \neq 0 : -\lambda^{-1}x \text{ is not q.r. in } \mathcal{A}\}$$

with $\{0\}$ added unless x is invertible in $\mathcal{A}$.

2.10. Proposition. [**M**, Proposition 2.8, p. 13] *If $\mathcal{A}$ is a locally m-convex algebra, then quasi-inversion is continuous on the set of q.r. elements.*

2.11. Definition. A *division algebra* is an algebra in which every nonzero element is invertible.

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2.12. Definitions. [**La**, Defintions 1.1.5, p. 7] Let $\mathcal{A}$ be a complex algebra and I an ideal of $\mathcal{A}$. Then I is called *regular* if the quotient algebra $\mathcal{A}/I$ is unital. i.e there exist $u \in \mathcal{A}$ such that two sets

$$\mathcal{A}(1-u) = \{x - xu : x \in \mathcal{A}\}$$

and

$$(1-u)\mathcal{A} = \{x - ux : x \in \mathcal{A}\}$$

are both contains in I. Such an element u is called an *identity modulo* I. The ideal I is said to be a *maximal regular ideal* if it is regular and also a maximal proper ideal.

2.13. Proposition. [**M**, Proposition 2.9, p. 13] *Let $\mathcal{A}$ be a locally m-convex algebra. Then*

(1) *If $x \in \mathcal{A}$, then $\sigma_{\mathcal{A}}(x)$ is not empty.*
(2) *If $\mathcal{A}$ is division algebra, then $\mathcal{A}$ is isomorphic to the algebra of complex numbers.*
(3) *If $\mathcal{A}$ is commutative, and if M is a closed, regular, maximal ideal in $\mathcal{A}$, then $\mathcal{A}/M$ is isomorphic to the algebra of complex numbers.*

Proof. (1) If $\mathcal{A}$ has no unit then $0 \in \sigma_{\mathcal{A}}(x)$. i.e. x is not invertible. Suppose x is invertible, then there exists $z + \lambda 1 \in \mathcal{A}$ such that $x \cdot (z + \lambda 1) = 1$. This gives $x \cdot z + x \cdot \lambda = 1$. But $x \cdot z + x \cdot \lambda \in \mathcal{A}$ therefore, $1 \in \mathcal{A}(e \in \mathcal{A}, 1 \in \mathcal{A}_e)$ which is a contradiction. So, x is not invertible. Hence, $0 \in \sigma_{\mathcal{A}}(x)$.
Suppose $\mathcal{A}$ is unital. Obviously $\overline{\mathcal{A}_i}$ is unital Banach algebra by Lemma 2.5. Now $\mathcal{A}$ is a locally m-convex algebra. Let $x \in \mathcal{A}$ and p_i denotes the seminorm on $\mathcal{A}$ generated by U_i. Let $\pi_i : \mathcal{A} \to \mathcal{A}/N_i(\mathcal{A}_i)$ by

$$\pi_i(x) = x + N_i.$$

This is a homomorphism from $\mathcal{A}$ onto $\mathcal{A}/N_i$. Let $\lambda \in \sigma_{\overline{\mathcal{A}_i}}(x + N_i)$. i.e. $x + N_i - \lambda$ is not invertible in $\mathcal{A}/N_i$. Suppose $x - \lambda$ is invertible in $\mathcal{A}$. Then, there exist $y \in \mathcal{A}$ such that

$$(x - \lambda)y = y(x - \lambda) = 1.$$

So that, $(x + N_i - \lambda)(y + N_i) = (y + N_i)(x + N_i - \lambda) = 1 + N_i$. Thus $\lambda \notin \sigma_{\overline{\mathcal{A}_i}}(x + N_i)$ which is a contradiction. Then $\lambda \in \sigma_{\mathcal{A}}(x)$. Therefore, $\sigma_{\overline{\mathcal{A}_i}}(x + N_i) \subset \sigma_{\mathcal{A}}(x)$. Here, $\sigma_{\overline{\mathcal{A}_i}}(x + N_i)$ is nonempty by Theorem **??**. Hence $\sigma_{\mathcal{A}}(x)$ cannot be empty.
(2) Let $\mathcal{A}$ be a division algebra and $x \in \mathcal{A}$. If possible that $x - \lambda \neq 0$ for any $\lambda \in \mathbb{C}$. Then $x - \lambda 1$ is invertible for every $\lambda \in \mathbb{C}$. This means that

$\sigma_{\mathcal{A}}(x)$ is an empty set which is a contradiction. Thus $\mathcal{A} = \{\lambda 1 : \lambda \in \mathbb{C}\}$. Consequently, $\lambda 1 \in \mathcal{A} \mapsto \lambda$ is an algebra isomorphism. Since every linear map on a finite dimensional normed linear space is continuous, we assert that this map is also a topological isomorphism. This completes the proof.
(3) Since M is a maximal regular ideal, $\mathcal{A}/M$ is division algebra [**La**, Theorem 3.1.1, p.65]. Hence,

$$\mathcal{A}/M \cong \mathbb{C}.$$

□

2.14. Definition. [**M**, p. 6] If E is a topological linear space, then E' denotes the space of *continuous linear functionals* on E.

2.15. Definition. [**M**, p. 6] If $\mathcal{A}$ is a topological algebra, then

$$\mathscr{M}(\mathcal{A}) = \{f \in \mathcal{A}' : f(xy) = f(x)f(y) \text{ for all } x, y \in \mathcal{A}\},$$

equipped with the relative topology induced on it by the topology $\sigma(\mathcal{A}', \mathcal{A})$ on $\mathcal{A}'$. ($\sigma(\mathcal{A}', \mathcal{A})$ denotes the week topology induced on $\mathcal{A}'$ by $\mathcal{A}$.) Furthermore, $\mathscr{M}^{-}(\mathcal{A})$ denotes the subspace (with the relative topology) of $\mathscr{M}(\mathcal{A})$ which we get by removing the zero functional from $\mathscr{M}(\mathcal{A})$.

2.16. Proposition. [**H**] *Let V be a vector space over a field F and $f : V \to F$ be a nonzero linear map. Then $W = \{v \in V : f(v) = 0\}$ is a maximal subspace of V.*

2.17. Proposition. *Let $\mathcal{A}$ be a topological algebra. Let $\psi \in \mathscr{M}^{-}(\mathcal{A})$. If ψ is nonzero, then $ker(\psi)$ is a closed, regular, maximal ideal of $\mathcal{A}$.*

Proof. $ker(\psi) = \{x \in \mathcal{A} : \psi(x) = 0\}$. Since ψ is linear, $ker(\psi)$ is a subspace of $\mathcal{A}$. Now let $x \in ker(\psi)$ and $a \in \mathcal{A}$. Then

$$\psi(ax) = \psi(a)\psi(x) = 0$$

and

$$\psi(xa) = \psi(x)\psi(a) = 0.$$

Therefore $ker(\psi)$ is an ideal of $\mathcal{A}$. Also,

$$\psi^{-1}(\{0\}) = \{x \in \mathcal{A} : \psi(x) = 0\} = ker(\psi).$$

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Since $\{0\}$ is closed, inverse image of closed set is closed. Thus $ker(\psi)$ is closed. Now, to verify that $ker(\psi)$ is regular. Choose $u \in \mathcal{A}$ such that $\psi(u) = 1$. Then for any $x \in \mathcal{A}$,

$$\begin{aligned}\psi(ux - x) &= \psi(u)\psi(x) - \psi(x)\\ &= \psi(x) - \psi(x)\\ &= 0.\end{aligned}$$

So, $ux - x \in ker(\psi)$. Thus, $ker(\psi)$ is regular. Now, by Proposition 2.16 and the fact $\psi \neq 0$, $ker(\psi)$ is maximal among the family of proper subspaces of $\mathcal{A}$. Hence it is also maximal among the family of ideals of $\mathcal{A}$. This completes the proof. □

2.18. Corollary. [**M**, Corollary 2.10, p. 14] *If $\mathcal{A}$ is a commutative, locally m-convex algebra and $m(\mathcal{A})$ be the collection of closed, regular, maximal ideals in $\mathcal{A}$, then there is a 1-1 correspondence between $\mathscr{M}^-(\mathcal{A})$ and $m(\mathcal{A})$ in $\mathcal{A}$, such that every $f \in \mathscr{M}^-(\mathcal{A})$ corresponds to its kernel.*

Proof. Let $\mathcal{A}$ be a commutative locally m-convex algebra. Let M be a closed regular, maximal ideal in $\mathcal{A}$. By Proposition 2.13, $\mathcal{A}/M \cong \mathbb{C}$ with the isometric isomorphism $\phi_{\mathcal{A}/M} : \mathcal{A}/M \to \mathbb{C}$. Now, consider the homomorphism $\pi_M : \mathcal{A} \to \mathcal{A}/M$ defined as $\pi_M(x) = x + M, (x \in \mathcal{A})$. We observe that

$$\begin{aligned}ker(\pi_M) &= \{x \in \mathcal{A} : \pi_M(x) = 0 + M\}\\ &= \{x \in \mathcal{A} : x + M = 0 + M\}\\ &= \{x \in \mathcal{A} : x \in M\}\\ &= M.\end{aligned}$$

Consequently, $\psi_M = \phi_{\mathcal{A}/M} \circ \pi_M$ is a complex homomorphism from $\mathcal{A}$ to $\mathbb{C}$. Thus, $\psi_M \in \mathscr{M}^-(\mathcal{A})$. Also,

$$\begin{aligned}ker(\psi_M) &= \{x \in \mathcal{A} : \psi_M(x) = 0\}\\ &= \{x \in \mathcal{A} : (\phi_{\mathcal{A}/M} \circ \pi_M)(x) = 0\}\\ &= \{x \in \mathcal{A} : \phi_{\mathcal{A}/M}(\pi_M(x)) = 0\}\\ &= \{x \in \mathcal{A} : \pi_{\mathcal{A}/M}(x) = 0 + M\}\\ &= ker(\pi_M)\\ &= M.\end{aligned}$$

Thus, $ker(\phi_{\mathcal{A}/M} \circ \pi_M) = M$. Since M is proper, $\psi_M \neq 0$. Suppose, another $\psi \in \mathscr{M}^-(\mathcal{A})$ such that $ker(\psi) = M$. So, for any $y \in \mathcal{A}$

$$\begin{aligned} &\psi(y - \psi(y)1) = 0 \\ &\Rightarrow \psi_M(y - \psi(y)1) = 0 \\ &\Rightarrow \psi_M(y) - \psi(y)\psi_M(1) = 0 \\ &\Rightarrow \psi_M(y) = \psi(y) \end{aligned}$$

Thus, there exists unique $\psi_M \in \mathscr{M}^-(\mathcal{A})$ such that $M = ker(\psi_M)$. Now, define

$$\mathscr{T} : \mathscr{M}^-(\mathcal{A}) \to m(\mathcal{A})$$

by

$$\mathscr{T}(\psi) = ker(\psi), \quad (\psi \in \mathscr{M}^-(\mathcal{A})).$$

By Proposition 2.17, $ker(\psi) \in m(\mathcal{A})$. Let $\psi_1, \psi_2 \in \mathscr{M}^-(\mathcal{A})$, then

$$\mathscr{T}(\psi_1) = \mathscr{T}(\psi_2) \Rightarrow ker(\psi_1) = ker(\psi_2) \Rightarrow \psi_1 = \psi_2.$$

Also, for any $M \in m(\mathcal{A})$ there exists unique $\psi \in \mathscr{M}^-(\mathcal{A})$ such that $ker(\psi) = M$. Thus $\mathscr{T}$ is onto. This completes the proof. □

2.19. Proposition. [**Ros**, Proposition 3.3, p.13] *Let $\mathcal{A}$ be a commutative locally m-convex algebra with identity. Then $\mathscr{M}^-(\mathcal{A})$ is not empty.*

Proof. Let p be a continuous submultiplicative seminorm on $\mathcal{A}$. Then $\mathcal{A}_p = \mathcal{A}/N_p$ is a commutative normed algebra with identity under the norm induced by p, and so its completion $\overline{\mathcal{A}_p}$ is a Banach algebra. Let $\phi \in \mathscr{M}^-(\overline{\mathcal{A}_p}) \neq \emptyset$ and let $\pi : \mathcal{A} \to \mathcal{A}_p$ be the canonical homomorphism. Since π is continuous $\phi \circ \pi \in \mathscr{M}^-(\mathcal{A}) \neq \emptyset$. □

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