Heterogeneous Arrival Fluid Model Driven by an M/M/1 Queue with Exponential Vacation Subject to Catastrophe

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Abstract- **Fluid model is suitable for modeling traffic network where individual arrival has less impact on the performance of the network. For evaluating the performance measure, it is important to obtain information about the buffer content. This paper studies a fluid model driven by an M/M/1 queue with heterogeneous arrival and exponential vacation subject to catastrophe under steady state conditions. Based on that explicit expression for the Laplace transform of the stationary buffer content distribution with the minimal positive solution to a crucial quadratic equation is obtained .Then the performance measure – mean of the buffer content which is dependent on vacation parameter is obtained .Finally, the parameters effect on the mean buffer content is also illustrated by sensitivity analysis.**

Keywords- Fluid queue, Heterogeneous arrival, Exponential vacation, Catastrophe and Buffer Content.

I. INTRODUCTION

A Fluid queue is a mathematical model used to describe fluid level in a reservoir subject to randomly determined periods of filling and emptying the system without interruption called a buffer, according to a randomly varying rate regulated by an external stochastic environment. Such fluid queues are used as a mathematical tool for modeling, for example, to approximate discrete models, model the spread of wildfires in ruin theory and to model high speed data networks, a router, computer networks including call admission control, traffic shaping and modeling of TCP and production and inventory systems. Readers may refer to Anick, Sondhi and Mitra [1] for more details.

There have been many studies on fluid queue driven by Markovian environment and fluid queue driven by a classical queueing system (without vacation). M. F. Neuts [2] analyzed on structured stochastic matrices of M/G/I type and their applications. According to Neuts, we should always take the minimal positive solution to the crucial quadratic equation. Virtamo and Norros [3] proposed a spectral decomposition method. Adan. I, Resing. J [4] studied on a simple analysis of a fluid queue and expressed stationary buffer content. A complete survey on fluid queue with finite state Markovian environment was given by V.G. Kulkarni[5]. Sericola and Tuffin[6] derived a new approach for the computation of stationary buffer content. Sericola and Barbot[7] discussed on the state of M/M/1 queue and obtained expression for the joint stationary distribution of the buffer content. Quan Li, Liu and Shang[8] explained that the Laplace transform of the distribution of the stationary buffer content is expressed through the minimal positive solution to a crucial equation, similar to the fundamental equation satisfied by the busy period of an M/G/1 queue. Fuwei Wang, Mao and Tian [9] analyzed a fluid queue driven by an M/M/1 queue with single exponential vacation and obtained mean of the buffer content. Using spectral-analytic method, Mao, Wang and Tian [10] discussed M/M/1/N vacation queue and derived steady state distribution of buffer content and mean buffer content. Vijayshree. K.V, Anjuka.A [11] explained stationary analysis of fluid queue subject to catastrophe and subsequent repair. Ammar [12] discussed a fluid queue with multiple exponential vacations and derived an explicit expression for the stationary distribution of the buffer content. Sherif I. Ammar [13] considered the M/M/1 queues with disasters where the customers are impatient due to the absence of the server.

In this paper, we study a fluid model driven by M/M/1 queue with heterogeneous arrival and single exponential vacation subject to catastrophe. According to Neuts[2],we derive the Laplace transform of stationary buffer content with the minimal positive solution to a crucial quadratic equation. Then the performance measure- mean buffer content which is dependent on the vacation parameter is also obtained.

This paper is organized as follows: Descriptions of the model is given in section II .In section III, we set up system of steady state differential equations and derive Laplace transform of stationary buffer content. Hence, we obtain performance measure-mean buffer content. Finally, in section IV using sensitivity analysis, we present parameters effect on the mean buffer content.

II. MODEL DESCRIPTION

Assume a system consisting of M/M/1 fluid queue with heterogeneous arrival rate subject to catastrophe and buffer whose arrival rates are modulated by an single vacation queue. Following are the assumptions made to derive the process .Let $\lambda_{0,}$ λ_1 and λ_2 represent arrival rate during vacation, busy and idle respectively and μ denote service rate. Also consider Inter-arrival times of customers and service times of present fluid model are independent and identically distributed. The arrival parameters λ_0, λ_1, λ_2 follow poisson distribution and service rate μ follows exponential distribution. When the system is not empty (the system is busy in service time) catastrophe occurs at the service facility, according to poison process with rate ξ. The background process $\{N (t), t\geq 0\}$, where N(t) denotes number of customers in the system at time t. The vacation policy is the single vacation with exhaustive service and vacation time V is exponentially distributed with intensity parameter θ. Define J(t) as 0,1,2 when server is on vacation, busy or idle at time t. Here, we assume that arrival times, service time and vacation time are mutually independent with service discipline as First In First Out (FIFO).

The fluid from the first phase (i.e, fluid output of the M/M/1 vacation queue) goes to the second phase represents a buffer with a constant leak rate c. We always assume that service rate is greater than buffer, $\mu > c$. Otherwise, buffer would be empty. .Let C(t) denote buffer content at time t .The buffer content C(t) is a non-negative random variable. We note that both input and output from the buffer are on-off process with traffic intensity $\rho_1 = \lambda_1/\mu$. Furthermore, we assume that net input rate of fluid to the buffer is $r_b > 0$, that is the content of the buffer increases at the rate r during busy period of the server while the buffer decreases at the rate r_v , when the server is on vacation (ie) $r_v < 0$, when $\{J (t), t \geq 0\}$ is in busy state or not. Therefore when buffer is empty, content of the buffer cannot decrease. It is obvious that $r_v = -c$, $r_b = \mu - c$ implies $\mu = r_b - r_v$. When the server is busy in service time, the occurrence of catastrophe annihilates the entire system with rate ξ. Then the process $\{N (t), J (t), t\geq 0\}$ is a Markovian process with state space is given by,

$$\Omega = \{(k, 0), k=0, 1, 2\ldots\ldots\} \cup \{(k, 1), k=1,2\ldots\ldots\}$$

Let (N, J) denote stationary random vector of $\{N(t), J(t), t\geq 0\}$ whose stable probability distribution is $F_{kj} = P\{N=k, J=j\}$. When the process is stable, we write $F_{kj}(w) = \lim_{t\to\infty} F_{kj}(t, w)$ which are independent of the initial state of process. Hence, we assume that stability conditions are satisfied throughout the analysis.

III. MODEL EQUATIONS AND THE SOLUTION

Theorem 1: The stable joint probability distribution functions are satisfied with following system of differential equations

$$\frac{r_v dF_{00}(w)}{dw} = -(\lambda_0 + \theta)F_{00}(w) + \mu F_{11}(w) \quad (1)$$

$$\frac{r_v dF_{02}(w)}{dw} = \theta F_{00}(w) - \lambda_2 F_{02}(w) \quad (2)$$

$$\frac{r_v dF_{k0}(w)}{dw} = \lambda_0 F_{k-1,0}(w) - (\lambda_0 + \theta)F_{k0}(w), \quad k\geq 1 \quad (3)$$

$$\frac{r_b dF_{11}(w)}{dw} = \lambda_2 F_{02}(w) + \theta F_{10}(w) - (\lambda_1 + \mu + \xi)F_{11}(w) + \mu F_{21}(w) \quad (4)$$

$$\frac{r_b dF_{k1}(w)}{dw} = \theta F_{k0}(w) - (\lambda_1 + \mu + \xi)F_{k1}(w) + \mu F_{k+1,1}(w) + (\lambda_1 + \xi)F_{k-1,1}(w), \quad k \geq 2 \quad (5)$$

With the boundary conditions as specified by Mao, Wang and Tian [9] we have,

$$F_{02}(0) = (1+\frac{\lambda_2}{r_v})\frac{\theta^2}{\lambda_2^2+\lambda_2\theta+\theta^2} \quad (6)$$

$$F_{k0}(0) = [\frac{\lambda_0}{\lambda_0+\theta}]^k \frac{\lambda_0\theta}{\lambda_0^2+\lambda_0\theta+\theta^2}[1+\frac{\lambda_0}{r_v}],\ k \geq 0 \quad (7)$$

$$F_{k1}(0) = 0;\ k \geq 1 \quad (8)$$

Remark 1 : Considering equations (6) and (7), by direct calculation, we obtain

$$P\{C=0\} = [1+\frac{\lambda_0}{r_v}]\frac{\lambda_0^2+\lambda_0\theta}{\lambda_0^2+\lambda_0\theta+\theta^2} + [1+\frac{\lambda_2}{r_v}]\frac{\theta^2}{\lambda_2^2+\lambda_2\theta+\theta^2}$$

The above result can also be derived with the Law of Conservation. To derive solution to above system of differential equation , we write Laplace transform of functions $F_{kj}(t,w), ((k,j) \in \Omega)$ as $\hat{F}_{kj}(s) = \int_0^{+\infty} e^{-sw} F_{kj}(w)dw$.

Taking Laplace transform on both sides of the equation (1) to (5) and using equation (6) to (8), we obtain

$$(sr_v+\lambda_0+\theta)\hat{F}_{00}(s) = \frac{\lambda_0\theta}{\lambda_0^2+\lambda_0\theta+\theta^2}[\lambda_0+r_v] + \mu\hat{F}_{11}(s) \quad (9)$$

$$(sr_v+\lambda_2)\hat{F}_{02}(s) = \theta\hat{F}_{00}(s) + (\frac{\lambda_2\theta^2+\theta^2 r_v}{\lambda_2^2+\lambda_2\theta+\theta^2}) \quad (10)$$

$$(sr_v+\lambda_0+\theta)\hat{F}_{k0}(s) = \frac{(\lambda_0+r_v)\lambda_0\theta}{\lambda_0^2+\lambda_0\theta+\theta^2}[\frac{\lambda_0}{\lambda_0+\theta}]^k + \lambda_0\hat{F}_{k-1,0}(s), \quad k \geq 1 \quad (11)$$

$$(sr_b+\lambda_1+\mu+\xi)\hat{F}_{11}(s) = \lambda_2\hat{F}_{02}(s) + \theta\hat{F}_{10}(s) + \mu\hat{F}_{21}(s) \quad (12)$$

$$(sr_b+\lambda_1+\mu+\xi)\hat{F}_{k1}(s) = \theta\hat{F}_{k0}(s) + \mu\hat{F}_{k+1,1}(s) + (\lambda_1+\xi)\hat{F}_{k-1,1}(s), \quad k \geq 2 \quad (13)$$

By letting ,

$$F_0^*(s,z) = \sum_{k=1}^{\infty}\hat{F}_{k0}(s)z^k \quad \text{and} \quad F_1^*(s,z) = \sum_{k=1}^{\infty}\hat{F}_{k1}(s)z^k \quad (14)$$

Then from equation (11), we have

$$F_0^*(s,z) = \frac{\lambda_0 z\hat{F}_{00}(s)}{sr_v+\lambda_0(1-z)+\theta} + \frac{(\lambda_0+r_v)\lambda_0\theta}{\lambda_0^2+\lambda_0\theta+\theta^2}[\frac{\lambda_0 z}{(\lambda_0(1-z)+\theta)(sr_v+\lambda_0(1-z)+\theta)}] \quad (15)$$

By considering equations (9) and (10), which follow from the equations (12) and (13) we observe that,

$$[sr_b+\lambda_1(1-z)+\mu(1-\frac{1}{z})+\xi(1-z)]F_1^*(s,z) = [\frac{(\lambda_0+r_v)\lambda_0\theta}{\lambda_0^2+\lambda_0\theta+\theta^2}[1+\frac{\lambda_0\theta z}{(\lambda_0(1-z)+\theta)(sr_v+\lambda_0(1-z)+\theta}]$$
$$+\frac{\lambda_2 z\theta^2(\lambda_2+r_v)}{(\lambda_2^2+\lambda_2\theta+\theta^2)(sr_v+\lambda_2)} - [(sr_v+\lambda_0+\theta)$$
$$-\frac{\lambda_2\theta z}{sr_v+\lambda_2} - \frac{\lambda_0\theta z}{sr_v+\lambda_0(1-z)+\theta}]\hat{F}_{00}(s) \quad (16)$$

By using probability generating function (14), it is clear that $F_1^*(s,z)$ is analytic for the variable z. Hence, if $z_0(s)$ is a solution to the equation $sr_b + \lambda_1(1-z) + \xi(1-z) + \mu(1-(1/z))$ it must be the solution to equation (16).According to Neuts [2] , we must consider minimal positive solution to crucial quadratic equation. Let $z_0(s)$ be minimal positive solution to the equation $sr_b + \lambda_1(1-z) + \xi(1-z) = \mu((1/z)-1)$.

That is , $z_0(s) = \dfrac{sr_b + \lambda_1 + \mu + \xi - \eta}{2(\lambda_1 + \xi)}$ where $\eta = \sqrt{sr_b + \lambda_1 + \mu + \xi - 4(\lambda_1 + \xi)\mu}$

Hence we get,

$$\hat{F}_{00}(s) = \frac{\begin{array}{l}\{(\lambda_0(1-z_0)+\theta)(sr_v+\lambda_0(1-z_0)+\theta)[(\lambda_0+r_v)\lambda_0\theta(\lambda_2^2+\lambda_2\theta+\theta^2)(sr_v+\lambda_2)\\ +\lambda_2\theta^2 z_0(s)(\lambda_2+r_v)(\lambda_0^2+\lambda_0\theta+\theta^2)]\}\\ +[(\lambda_0\theta)^2 z_0(\lambda_0+r_v)(\lambda_2^2+\lambda_2\theta+\theta^2)(sr_v+\lambda_2)]\end{array}}{\begin{array}{l}(\lambda_0^2+\lambda_0\theta+\theta^2)(\lambda_2^2+\lambda_2\theta+\theta^2)(\lambda_0(1-z_0)+\theta)\\ \{(sr_v+\lambda_0(1-z_0)+\theta)[(sr_v+\lambda_0+\theta)(sr_v+\lambda_2)-\lambda_2\theta z_0]-\lambda_0\theta z_0(s)(sr_v+\lambda_2)\}\end{array}} \qquad (17)$$

Remark 2: When vacation time V is zero, considering equation (10) and with law of L' Hospital, we obtain

$$\lim_{\theta\to\infty} \hat{F}_{02}(s) = \frac{(\lambda_2 + r_v)}{[sr_v + \lambda_2(1 - z_0(s))]}$$

which is similar to the equation in Li, Liu, Shang [8]. Here, our result coincides with that of system without vacation. Similarly, we obtain $\lim_{\theta\to\infty} \hat{F}_{00}(s) = 0$. It is obvious that, once the system is empty, server will be in idle state, as in case of classical queue system without vacation.

The Steady State Distribution and Mean of Buffer Content

Let $H(w) = \lim_{t\to\infty} p\{C(t) \le w\}$ and denote $\bar{H}(w) = 1 - H(w)$ and $\hat{H}(s) = \int_0^{+\infty} e^{-sw} H(w)dw$

Theorem 2: The Laplace transform of the distribution of stationary buffer content for the stable fluid queue is given by

$$\begin{aligned}\hat{H}(s) = {} & [1 + \frac{\theta}{sr_v + \lambda_2}(1 + \frac{\lambda_2}{sr_b}) + \frac{\lambda_0}{sr_v + \theta}(1 + \frac{\theta}{sr_b}) - (\frac{sr_v + \lambda_0 + \theta}{sr_b})]\hat{F}_{00}(s)\\ & + \frac{(\lambda_0 + r_v)\lambda_0}{\lambda_0^2 + \lambda_0\theta + \theta^2}[\frac{\lambda_0}{sr_v + \theta}(1 + \frac{\theta}{sr_b}) + \frac{\theta}{sr_b}] + \frac{(\lambda_2 + r_v)\theta^2}{(\lambda_2^2 + \lambda_2\theta + \theta^2)(sr_v + \lambda_2)}[1 + \frac{\lambda_2}{sr_b}]\end{aligned}$$

Proof: By considering the total probability $\hat{H}(s) = \hat{F}_{00}(s) + \hat{F}_{02}(s) + F_0^*(s,1) + F_1^*(s,1)$ .Using equations (10), (15),(16) & (17) , we obtain the above result.

Remark 3: When vacation time V tends to zero, considering Theorem 2 and thereby applying the Law of L' Hospital

We obtain,

$$\hat{H}(s) = \frac{\lambda_2 + r_v}{sr_b}[\frac{(sr_b + \lambda_2(1 - z_0(s))}{(sr_v + \lambda_2(1 - z_0(s))}]$$

Theorem 3: Mean of stationary buffer content for stable fluid is given by

$$E(C) = \frac{(\lambda_1 + \xi) r_b}{(\lambda_1 + \xi - \mu)\left[\frac{(\lambda_0^2 + \lambda_0 \theta)(\lambda_0 + r_v)}{\lambda_0^2 + \lambda_0 \theta + \theta^2} + \frac{\theta^2 (\lambda_2 + r_v)}{\lambda_2^2 + \lambda_2 \theta + \theta^2}\right]} \qquad (18)$$

Proof: Let $\bar{H}(w) = 1 - H(w)$ and $\hat{H}(s) = \int_0^{+\infty} e^{-sw} H(w)\,dw$

We know that, $E(c) = \int_0^{+\infty} \bar{H}(w)\,dw = \lim_{s \to 0^+} \left[\frac{1}{s} - \hat{H}(s)\right]$

Considering Theorem 2 and $(r_b - r_v) = \mu$, and thereby applying Laplace transform formula and Law of L' Hospital, we obtain mean of stationary buffer content.

IV. NUMERICAL ANALYSIS

To demonstrate the applicability of the result, we illustrate graph to analyze the performance measure- mean buffer content of the system. We demonstrate the change of mean buffer content E(c) with different values of ρ_1 and net input rate r_v.The arrival rate during idle, busy and vacation period $(\lambda_2, \lambda_1, \lambda_0)$ is taken as 0.1,1 and 1.3. As illustrated in fig.1 for fixed ρ_1, ξ and θ, E(c) increases when absolute value of r_v decreases.Whereas, for fixed r_v, E(c)decreases when value of ρ_1 increases. Therefore , when ρ_1<0.1, the shape of E(c) does not show significant change.

Effect of ρ1 and rv on Mean Buffer Content E(C)

rv \ ρ1	0.5	0.33	0.1	0.01
-1.9	0.11355	0.628851	1.012852	1.114033
-1.8	0.255541	0.770229	1.153772	1.254833
-1.7	0.438191	0.952089	1.335044	1.43595
-1.6	0.68188	1.194725	1.576895	1.677594
-1.5	1.023345	1.534714	1.915784	2.016194
-1.4	1.536201	2.045353	2.424771	2.524745
-1.3	2.39272	2.898169	3.274828	3.374076
-1.2	4.112385	4.610401	4.981521	5.079308
-1.1	9.325022	9.800506	10.15483	10.2482

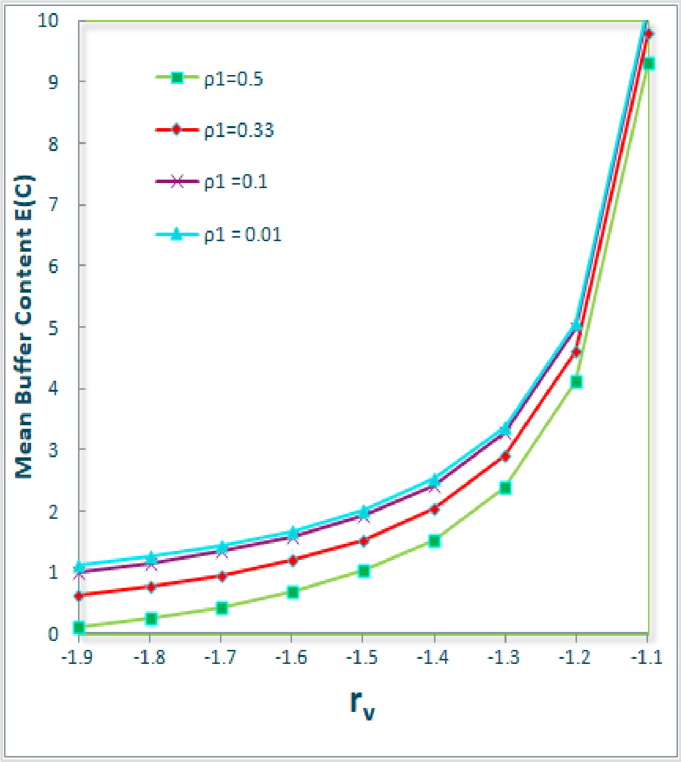

Fig.1 Effect of ρ_1 and r_v on Mean Buffer Content E(C)

Further, the effect of both busy period arrival rate (λ_1) and net input rate (r_v) on the mean buffer content is presented in fig.2. The arrival rate during idle and vacation period (λ_2 , λ_0) are taken as 0.1 and 1.03. As illustrated in fig.2, for fixed ξ and θ, E(C) increases along with increase of λ_1 and decrease with r_v. We also observe that E(C) decreases along with absolute values of r_v, for fixed values of λ_1 and ξ. Hence we find that that the increasing and decreasing in the values of mean buffer content E(C) are due to the variation of the busy period arrival rate (λ_1) and net input rate (r_v). This agrees with practical cases also.

Effect of λ 1 and rv on Mean Buffer Content E(C)					
rv \ λ 1	0	1	2	3	4
-6	0.000697	0.095533	0.27186	0.637785	1.574792
-6.1	0.000668	0.091206	0.258091	0.599638	1.450141
-6.2	0.00064	0.087055	0.245027	0.56412	1.338607
-6.3	0.000613	0.08307	0.232617	0.53097	1.238225
-6.4	0.000587	0.079241	0.220811	0.499958	1.147402
-6.5	0.000561	0.07556	0.209568	0.470884	1.064833
-6.6	0.000537	0.072017	0.198848	0.443571	0.989443
-6.7	0.000513	0.068605	0.188614	0.417865	0.920334
-6.8	0.00049	0.065318	0.178835	0.393628	0.856753
-6.9	0.000467	0.062148	0.169482	0.370736	0.798063
-7	0.000445	0.059088	0.160526	0.349082	0.743719
-7.1	0.000424	0.056135	0.151943	0.328568	0.693256
-7.2	0.000404	0.053281	0.143711	0.309105	0.646273
-7.3	0.000384	0.050522	0.135807	0.290615	0.602421
-7.4	0.000364	0.047854	0.128214	0.273027	0.561399
-7.5	0.000346	0.045272	0.120912	0.256276	0.52294
-7.6	0.000327	0.042772	0.113886	0.240305	0.486811
-7.7	0.000309	0.040349	0.10712	0.225059	0.452808
-7.8	0.000292	0.038002	0.100601	0.210491	0.420747
-7.9	0.000275	0.035725	0.094313	0.196556	0.390467
-8	0.000259	0.033516	0.088247	0.183214	0.361824

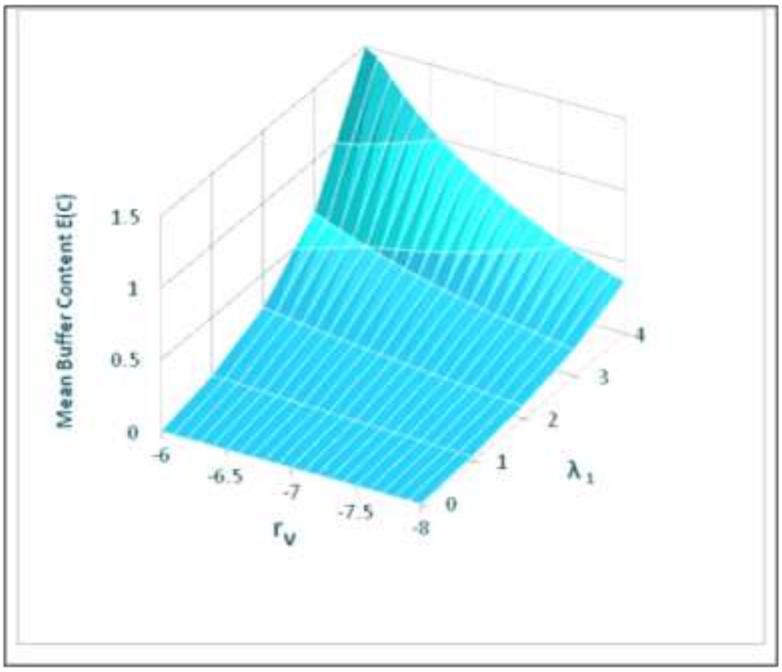

Fig.2 Effect of λ_1 and r_v on Mean Buffer Content E(C)

Conclusion: In this paper, we studied heterogeneous arrival fluid model driven by an M/M/1 queue with exponential vacation subject to catastrophe. We set up a system of differential equation to derive the Laplace transform of steady state distribution and concisely express steady state distribution of the buffer content through the minimal positive solution to a crucial quadratic equation. Finally, we calculate the performance measure of mean of the stable buffer content depending on vacation parameter θ. The study can be further extended to analyse heterogeneous arrival M/M/1 queue with vacation subject to catastrophe and balking.

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